Noether's problem

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Colloquium
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Symmetric polynomials

$x + y$

xy

$x^2 + y^2$

$x^2y + xy^2$

$x^3 + y^3$

$x^4y^4 + 2x^5y^2 + 2x^2y^5$

Symmetric polynomials

$x + y$

xy

$x^2 + y^2 = (x + y)^2 - 2xy$

$x^2y + xy^2 = (x + y)xy$

$x^3 + y^3 = (x + y)^3 - 3x^2y - 3xy^2$

$x^4y^4 + 2x^5y^2 + 2x^2y^5 = (xy)^4 + 2(xy)^2(x^3 + y^3)$

Symmetric polynomials

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Symmetric polynomials

$x + y = \sigma_1$

$xy = \sigma_2$

$x^2 + y^2 = \sigma_1^2 - 2\sigma_2$

$x^2y + xy^2 = \sigma_1\sigma_2$

$x^3 + y^3 = \sigma_1^3 - 3\sigma_1\sigma_2$

$x^4y^4 + 2x^5y^2 + 2x^2y^5 = \sigma_2^4 + 2\sigma_1^3\sigma_2^2 - 6\sigma_1\sigma_2^3$

Fundamental Theorem

Newton 1665, Waring 1770, Gauss 1815

Theorem. Any symmetric polynomial in variables $x_1, \ldots, x_n$ can be uniquely expressed as a polynomial in the *elementary symmetric polynomials*

$$\sigma_1 = x_1 + x_2 + \cdots + x_n$$

$$\sigma_2 = x_1 x_2 + x_1 x_3 + \cdots + x_{n-1} x_n$$

$$\vdots$$

$$\sigma_k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k}$$

$$\vdots$$

$$\sigma_n = x_1 x_2 \cdots x_n$$

Newton–Girard formulas

Girard 1629, Newton 1666

$p_k = x_1^k + x_2^k + \cdots + x_n^k$ power sums

$$\begin{aligned} p_1 &= \sigma_1 \\ p_2 &= \sigma_1^2 - 2\sigma_2 \\ p_3 &= \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3 \\ p_4 &= \sigma_1^4 - 4\sigma_1^2\sigma_2 + 2\sigma_2^2 + 4\sigma_1\sigma_3 - 4\sigma_4 \end{aligned}$$

$$p_k = \det \begin{pmatrix} \sigma_1 & 1 & 0 & \cdots & 0 \\ 2\sigma_2 & \sigma_1 & 1 & \cdots & 0 \\ 3\sigma_3 & \sigma_2 & \sigma_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ k\sigma_k & \sigma_{k-1} & \sigma_{k-2} & \cdots & \sigma_1 \end{pmatrix}$$

Vieta'a formula

$\alpha_1, \alpha_2, \ldots, \alpha_n$ roots of monic polynomial $f(x)$

$$f(x) = x^n - \sigma_1 x^{n-1} + \sigma_2 x^{n-1} - \cdots + (-1)^n \sigma_n$$

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Example. $2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, \ldots$

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Example. $2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, \dots$

$$L_n = L_{n-1} + L_{n-2} \text{ Lucas numbers}$$

$$\phi = \frac{1+\sqrt{5}}{2}, \quad \bar{\phi} = \frac{1-\sqrt{5}}{2} \quad \text{roots of } x^2 - x - 1$$

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$$L_n = \phi^n + \bar{\phi}^n = \det \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ -2 & 1 & 1 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \ddots & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 & 1 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}$$

Additive symmetric polynomial bases

partition $\lambda = (\lambda_1, \dots, \lambda_r)$ of n

monomial symmetric polynomials

$$m_\lambda = \sum_{\tau \in S_r} x_{\tau(1)}^{\lambda_1} \cdots x_{\tau(r)}^{\lambda_r}$$

Schur polynomials

$$s_\lambda = \sum_{T \text{ tableau } \lambda} x_1^{t_1} \cdots x_n^{t_n}$$

Algebra

S_n symmetric group of permutations of $\{1, 2, \ldots, n\}$

$\mathbb{C}[x_1, x_2, \ldots, x_n]$ ring of polynomials in variables $x_1, x_2, \ldots, x_n$

S_n acts on $\mathbb{C}[x_1, x_2, \ldots, x_n]$ by permuting the variables

Example. $(1234) \cdot (x_1 x_2 + x_3^2) = x_2 x_3 + x_4^2$

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Fundamental Theorem.
$$\mathbb{C}[x_1, x_2, \dots, x_n]^{S_n} = \mathbb{C}[\sigma_1, \sigma_2, \dots, \sigma_n]$$
$$\mathbb{C}(x_1, x_2, \dots, x_n)^{S_n} = \mathbb{C}(\sigma_1, \sigma_2, \dots, \sigma_n)$$

Ring of symmetric polynomials
Field of symmetric rational functions

Algebra

Fundamental Theorem. Any symmetric polynomial can be uniquely expressed as a polynomial in the elementary symmetric polynomials.

$$\begin{aligned}\text{uniqueness} &\Longleftrightarrow \sigma_1, \sigma_2, \ldots, \sigma_n \text{ algebraically independent} \\ &\Longleftrightarrow \mathbb{C}(x_1, x_2, \ldots, x_n)^{S_n} = \mathbb{C}(\sigma_1, \sigma_2, \ldots, \sigma_n) \\ &\ \text{purely transcendental over } \mathbb{C}\end{aligned}$$

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example.

1   2   3   4   5   6   7   8   9

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example.

4 7 8 5 3 1 9 2 6

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Bubble sort

4 7 5 8 3 1 9 2 6 1

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Bubble sort

4 7 5 3 8 1 9 2 6 2

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Bubble sort

4 7 5 3 1 8 9 2 6 3

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Bubble sort

4 7 5 3 1 8 2 9 6 4

Alternating polynomials

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Example. Bubble sort

4 7 5 3 1 8 2 6 9 5

Alternating polynomials

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Example. Bubble sort

4 5 7 3 1 8 2 6 9 6

Alternating polynomials

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Example. Bubble sort

4 5 3 7 1 8 2 6 9 7

Alternating polynomials

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Example. Bubble sort

4 5 3 1 7 8 2 6 9 8

Alternating polynomials

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Example. Bubble sort

4 5 3 1 7 2 8 6 9 9

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Bubble sort

4 5 3 1 7 2 6 8 9 10

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Bubble sort

4 3 5 1 7 2 6 8 9 11

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Bubble sort

4 3 1 5 7 2 6 8 9 12

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Bubble sort

4 3 1 5 2 7 6 8 9 13

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Bubble sort

4 3 1 5 2 6 7 8 9 14

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Bubble sort

3 4 1 5 2 6 7 8 9 15

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Bubble sort

3 1 4 5 2 6 7 8 9 16

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Bubble sort

3 1 4 2 5 6 7 8 9 17

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Bubble sort

1 3 4 2 5 6 7 8 9 18

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Bubble sort

1 3 2 4 5 6 7 8 9 19

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Bubble sort

1 2 3 4 5 6 7 8 9 20

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Bubble sort

1 2 3 4 5 6 7 8 9 20

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Quicksort

4 7 8 5 3 1 9 2 6

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Quicksort

4 2 8 5 3 1 9 7 6 1

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Quicksort

4 2 1 5 3 8 9 7 6

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Quicksort

4 2 1 3 5 8 9 7 6 3

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Quicksort

3   2   1   4   5   8   9   7   6   4

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Quicksort

3 1 2 4 5 8 9 7 6 5

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Quicksort

2 1 3 4 5 8 9 7 6 6

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Quicksort

1 2 3 4 5 8 9 7 6 7

Alternating polynomials

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Example. Quicksort

1 2 3 4 5 6 9 7 8 8

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Quicksort

1 2 3 4 5 6 8 7 9 9

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Quicksort

1 2 3 4 5 6 7 8 9 10

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Quicksort

1 2 3 4 5 6 7 8 9 10

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Another sort

4 7 8 5 3 1 9 2 6

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Another sort

$\color{red}{1}$ 7 8 5 3 $\color{red}{4}$ 9 2 6 1

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Another sort

1 2 8 5 3 4 9 7 6   2

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Another sort

1   2   3   5   8   4   9   7   6   3

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Another sort

1 2 3 4 8 5 9 7 6 4

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Another sort

1 2 3 4 5 8 9 7 6 5

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Another sort

1 2 3 4 5 6 9 7 8 6

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Another sort

1 2 3 4 5 6 7 9 8 7

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Another sort

1 2 3 4 5 6 7 8 9 8

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Another sort

1 2 3 4 5 6 7 8 9

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Another sort

4 7 8 5 3 1 9 2 6 even

Parity of the number of swaps is an invariant of a permutation

Alternating polynomials

$A_n \subset S_n$ subgroups of even permutations

Example. Another sort

4 7 8 5 3 1 9 2 6 even

Parity of the number of swaps is an invariant of a permutation

Definition. Ring of alternating polynomials $\mathbb{C}[x_1, x_2, \ldots, x_n]^{A_n}$
Field of alternating rational functions $\mathbb{C}(x_1, x_2, \ldots, x_n)^{A_n}$

Alternating polynomials

Vandermonde determinant

$$\delta = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & X_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^n & x_2^n & \cdots & x_n^n \end{pmatrix} = \prod_{i<j} (x_j - x_i)$$

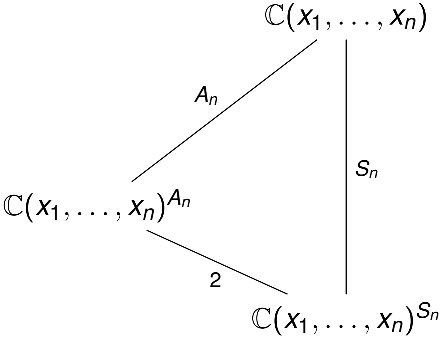
For any permutation $\tau \in S_n$

$$\tau(\delta) = \pm\delta \qquad \text{and} \qquad \tau(\delta) = \delta \iff \tau \in A_n$$

Discriminant

$$\delta^2 = \Delta = \prod_{i<j} (x_j - x_i)^2$$

Galois theory of alternating functions



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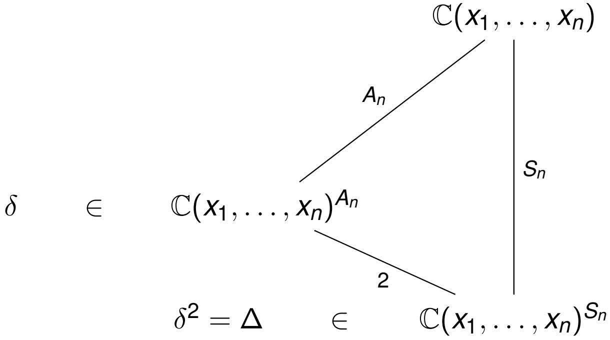

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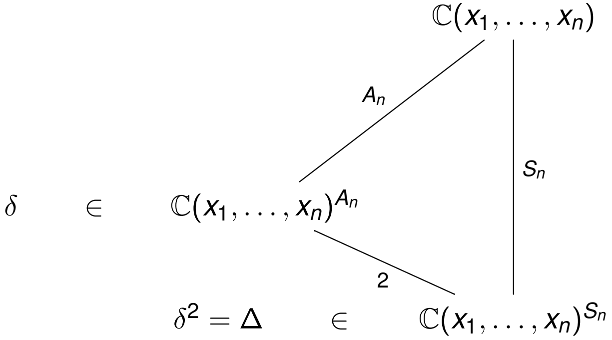

Theorem.

$$\mathbb{C}(x_1, \dots, x_n)^{A_n} = \mathbb{C}(\sigma_1, \dots, \sigma_n, \delta)$$
$$\mathbb{C}[x_1, \dots, x_n]^{A_n} = \mathbb{C}[\sigma_1, \dots, \sigma_n, \delta]$$

Elementary alternating functions?

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$$\mathbb{C}[x_1, \ldots, x_n]^{A_n} = \mathbb{C}[\sigma_1, \ldots, \sigma_n, \delta]$$

However, $\sigma_1, \ldots, \sigma_n, \delta$ are algebraically dependent!

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Question. Are there "elementary alternating functions" with $\mathbb{C}(x_1, \ldots, x_n)^{A_n} = \mathbb{C}(a_1, \ldots, a_n)$?

Elementary alternating functions?

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$$\mathbb{C}(x_1, \ldots, x_n)^{A_n} = \mathbb{C}(\sigma_1, \ldots, \sigma_n, \delta)$$
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However, $\sigma_1, \ldots, \sigma_n, \delta$ are algebraically dependent!

Question. Are there "elementary alternating functions" with $\mathbb{C}(x_1, \ldots, x_n)^{A_n} = \mathbb{C}(a_1, \ldots, a_n)$?

$\Longleftrightarrow$

Is $\mathbb{C}(x_1, \ldots, x_n)^{A_n}$ purely transcendental over $\mathbb{C}$?

Elementary alternating functions?

Theorem. $$\begin{aligned}\mathbb{C}(x_1,\ldots,x_n)^{A_n} &= \mathbb{C}(\sigma_1,\ldots,\sigma_n,\delta)\\ \mathbb{C}[x_1,\ldots,x_n]^{A_n} &= \mathbb{C}[\sigma_1,\ldots,\sigma_n,\delta]\end{aligned}$$

However, $\sigma_1,\ldots,\sigma_n,\delta$ are algebraically dependent!

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$\Longleftrightarrow$
Is $\mathbb{C}(x_1,\ldots,x_n)^{A_n}$ purely transcendental over $\mathbb{C}$?

Answer. Known for $n = 3, 4, 5$. Completely open for $n \geq 6$.

Example. $\mathbb{C}[x_1,x_2,x_3]^{A_3}$ generated by

$$x_1 + x_2 + x_3, \quad x_1x_2 + x_2x_3 + x_1x_3, \quad (x_1 + \omega x_2 + \omega^2 x_3)^3$$

Amalie Emmy Noether

1882–1935

Amalie Emmy Noether

1882–1935

Noether’s problem

Jahresbericht der Deutschen Mathematiker-Vereinigung 1913

Körper und Systeme rationaler Funktionen.

Von

Emmy Noether in Erlangen.

Die vorliegende Arbeit behandelt *Basisfragen bei beliebigen Systemen rationaler und ganzer rationaler Funktionen*; und zwar lassen die angewandten Methoden der Körpertheorie die Erledigung dieser Fragen bei *Körpern* aus rationalen Funktionen — rationalen Funktionenkörpern*) — als das Wesentliche erscheinen, während die Verallgemeinerung der Resultate auf beliebige Systeme sich als Folgerung ergibt.

Von Basisfragen bei allgemeinen Systemen ist bis jetzt nur die durch das Hilbertsche Theorem (Math. Ann. 36) gewährleistete Existenz der *Modulbasis* bekannt. Im folgenden wird die *Frage der rationalen Darstellbarkeit mit der Existenz der Rationalbasis für jedes beliebige System vollständig beantwortet* (§ 7); die Rationalbasis der *Körper* ergibt sich schon in § 4. Diese Existenz der Rationalbasis erlaubt es, durchweg von dem abstrakt definierten Körper oder System auszugehen und dadurch solche Schwierigkeiten zu vermeiden, die nur durch die spezielle Wahl der Rational-

Noether's problem

k field
G finite group
V finite dimensional representation
$k(V)$ field of rational functions on V

Problem. Is $k(V)^G$ purely transcendental over k?

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Theorem (Noether). Noether's problem $\Longrightarrow$ inverse Galois

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Theorem (Noether). Noether's problem $\Longrightarrow$ inverse Galois

Inverse Galois problem. Does every finite group occur as a Galois group over $\mathbb{Q}$?

Noether's problem

History of positive results

(Newton 1665) $G = S_n$ symmetric group over $\mathbb{Q}$

(Burnside 1908) $G = A_n$ alternating groups $n = 3, 4$ over $\mathbb{Q}(\zeta_3)$

(Fischer 1915) G abelian groups over $\mathbb{Q}(\zeta_{\mathrm{exp(G)}})$

(Noether 1918) $G \subseteq S_4$ over $\mathbb{Q}$

(Furtwängler 1925) $G \subset S_n$ solvable $n = 3, 5, 7, 11$ over $\mathbb{Q}$

(Gröbner 1934) Q_8 quaternion group of order 8 over $\mathbb{Q}$

(Maeda 1989) $G = A_5$ alternating group over $\mathbb{Q}$

Polynomials vs. rational functions

Theorem (Chevalley–Shephard–Todd).
$k[V]^G = k[b_1, \ldots, b_m] \Longleftrightarrow G$ is a pseudoreflection group

Examples. Over $\mathbb{C}$ they are classified into 37 families:
S_n, C_n, $C_n.A_4$, $C_n.A_5$, . . . , $W(E_6)$, $W(E_7)$, $W(E_8)$

However, A_n for $n \geq 4$ are not pseudoreflection groups over $\mathbb{C}$!

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Noether's problem

History for A_4

(Burnside 1908) $G = A_4$ by providing explicit functions

$$e_1 = x_1 + x_2 + x_3 + x_4$$
$$z_1 = x_1 + x_2 - x_3 - x_4$$
$$z_2 = x_1 - x_2 + x_3 - x_4$$
$$z_3 = x_1 - x_2 - x_3 + x_4$$

$$\mathbb{C}(x_1, x_2, x_3, x_4)^{A_4} = \mathbb{C}\left(e_1, \frac{z_1^2+z_2^2+z_3^2}{z_1 z_2 z_3}, \frac{z_1^4+z_2^4+z_3^4}{z_1 z_2 z_3}, \frac{(z_1^2+\zeta_3 z_2^2+\zeta_3^2 z_3^2)^3}{z_1 z_2 z_3}\right)$$

(Jacobson 1980s) Exercise in "Basic Algebra II, 1st ed"

(Hajja 1989) Did the exercise: *The alternating functions of three and of four variables*, Algebras Groups Geom. **6** (1), 49–54

(Jacobson 1989) Exercise in "Basic Algebra II, 2nd ed"

Noether's problem

History of negative results

(Swan 1969) $G = C_n$ cyclic group $n = 47, 113, 223$ over $\mathbb{Q}$

Problem (Steenrod 1960). Let G be a finite abelian group. Does there exist X a homology $K(G, n)$ with an action of $\mathrm{Aut}(G)$ inducing the natural action on $H_n(X, \mathbb{Z}) = G$?

(Lenstra 1974) $G = C_8$ over $\mathbb{Q}$

Grunwald-Wang Problem. Does there exist a number field $K/\mathbb{Q}$ with prescribed p-adic completions?

(Saltman 1984, Bogomolov 1987) $|G| = p^6$ over $\mathbb{C}$

Brauer group. Nontriviality of $\mathrm{Br}_{\mathrm{ur}}$ or torsion in H_3 of smooth model of $\widetilde{\mathbb{A}^n/G}$.

(Moravec 2012) $|G| = 3^5$ over $\mathbb{C}$

Computer. New computational tools for computing $\mathrm{Br}_{\mathrm{ur}}$.