# Cohomological invariants of line bundle-valued symmetric bilinear forms

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ABSTRACT

Cohomological invariants of line bundle-valued symmetric bilinear forms

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The object of this dissertation is to construct cohomological invariants for symmetric bilinear forms with values in a line bundle $\mathscr{L}$ on a scheme $X$. These generalize the classical Hasse-Witt (or Stiefel-Whitney) invariants when $\mathscr{L}$ is the trivial line bundle. In this case, Jardine computes the étale cohomology ring of the classifying scheme of the orthogonal group to define universal invariants. There is no comparable theory when $\mathscr{L}$ is not trivial. Our approach is to utilize coboundary maps on nonabelian cohomology sets arising from covers of the orthogonal similitude group scheme. A new feature of this construction is a four-fold cover of the orthogonal similitude group by the Clifford group which "interpolates" between the Kummer double cover of the multiplicative group and the classical spin cover of the orthogonal group. This four-fold cover allows us to define an analogue of the 2nd Hasse-Witt invariant for $\mathscr{L}$-valued forms.

As for calculating the new invariants, we provide explicit formulas in the cases of odd rank forms and $\mathscr{L}$-valued metabolic forms. We also relate the invariants to parametrizations of $\mathscr{L}$-valued forms arising from exceptional isomorphisms of algebraic groups. One interesting case concerns forms of rank 6 with trivial Arf invariant. These arise from the reduced pfaffian construction of Knus, Parimala, and Sridharan, applied to 2-torsion Azumaya algebras of degree 4. We relate the new invariant of a reduced pfaffian form to the class of the corresponding Azumaya algebra in a refined involutive Brauer group defined by Parimala and Srinivas.

# Contents

# Introduction

The object of this dissertation is the construction of cohomological invariants for symmetric bilinear forms with values in a line bundle $\mathscr{L}$ over an arbitrary scheme in which 2 is invertible. These generalize the classical Hasse-Witt (or Stiefel-Whitney) invariants when $\mathscr{L}$ is the trivial line bundle.

The analogue of the 1st Hasse-Witt invariant, or discriminant, for even rank $\mathscr{L}$-valued forms is already contained in Parimala/Sridharan [40]. We construct an analogue of the 2nd Hasse-Witt invariant for $\mathscr{L}$-valued forms of arbitrary rank. Such invariants, as in the classical case, find their usefulness both in classifying $\mathscr{L}$-valued forms and in the study of Brauer group classes of arithmetic significance.

Why have these cohomological invariants not been constructed before? When $\mathscr{L}$ is the trivial line bundle, one can use the work of Jardine [27] on the étale cohomology ring (with $\mathbb{Z}/2\mathbb{Z}$ coefficients) of the simplicial classifying scheme of the orthogonal group to define universal Hasse-Witt invariants. The orthogonal group scheme (of a standard sum-of-squares form) on $X$ is the base change of a smooth affine algebraic group scheme on $\operatorname{Spec}\mathbb{Z}[\frac{1}{2}]$ (even on $\operatorname{Spec}\mathbb{Z}$). This enables the étale cohomology ring of the simplicial classifying scheme to be computed–in analogy with the topological case–by utilizing the injectivity on cohomology of restriction to the "maximal torus" for the orthogonal group. When $\mathscr{L}$ is not trivial this approach breaks down. When $\mathscr{L}$ is not a square in the Picard group, there is no known calculation of the étale cohomology ring of the simplicial classifying scheme of the orthogonal group of an $\mathscr{L}$-valued form. The orthogonal group scheme of such an $\mathscr{L}$-valued form is *not* the base change of any group scheme on $\operatorname{Spec}\mathbb{Z}[\frac{1}{2}]$. It is an *outer form* of the standard orthogonal group, i.e. a twist by a cocycle of automorphisms that cannot be lifted to a cocycle of inner automorphisms. As Serre famously points out, the cohomology of a group has in general no relation to the cohomology of an outer form.† There is obviously much work to be done in this direction.

Below, we employ a different approach. While isometry classes of $\mathscr{L}$-valued forms are torsors for an outer form of the orthogonal group, their similarity classes are torsors for a standard (i.e. definable over $\operatorname{Spec}\mathbb{Z}[\frac{1}{2}]$) group of orthogonal similitudes. A recent calculation of Holla/Nitsure [25], [26] shows that (over $\operatorname{Spec}\mathbb{C}$) the $\mathbb{Z}/2\mathbb{Z}$-cohomology of the classifying space of the even rank orthogonal similitude group does not contain an element analogous to the universal 2nd Hasse-Witt invariant. We may interpret this as the nonexistence of a natural "pin" double cover of the even rank orthogonal similitude group. Our contribution is the realization that, while there

†After proving that the cohomology groups of inner forms are bijective, concerning outer forms [47, I, §5.5, Remarque] Serre warns, "Par contre, $H^1(G, {}_aA)$ *n'a en général aucune relation avec* $H^1(G, A)$." Then later in [47, I, §5.7, Remarque 1], Serre feels the need to reiterate himself, "Ici encore, il est faux en général que $H^1(G, {}_cB)$ soit en correspondance bijective avec $H^1(G, B)$." This is indeed an important point.

is no natural double cover, there is a natural four-fold cover of the orthogonal similitude group by the Clifford group, which "interpolates" between the Kummer double cover of the multiplicative group and the classical pin cover of the orthogonal group (see §2.1.3). The kernel $\boldsymbol{\kappa}$ of this cover is in general a nonconstant group scheme and it defines a new cohomological invariant in $H^2_{\text{ét}}(X, \boldsymbol{\kappa})$ for similarity classes of $\mathscr{L}$-valued symmetric bilinear forms that "interpolates" between the classical 2nd Hasse-Witt invariant and the 1st Chern class modulo 2. The group scheme $\boldsymbol{\kappa}$ is locally isomorphic to $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$ when the rank is $\equiv 2, 3 \mod 4$ and to $\boldsymbol{\mu}_4$ when the rank is $\equiv 0, 1 \mod 4$ (see Propositions 2.8 and 2.11).

As for calculating this invariant, we provide explicit formulas for the general cases of forms of odd rank (see Theorem 2.19) and $\mathscr{L}$-valued metabolic forms (see Theorem 2.25). One interesting case concerns forms of rank 6 with trivial Arf invariant (see §3.1 and Theorem 3.5). These arise from the reduced pfaffian construction of Knus [31] and Knus/Parimala/Sridharan [32] applied to 2-torsion Azumaya algebras of degree 4. We relate the new invariant of a reduced pfaffian form to the class of the corresponding Azumaya algebra in the involutive Brauer group of Parimala/Srinivas [41].

There are two main ingredients in these calculations. First, we utilize the implications on categories of torsors of well-chosen commutative diagrams relating Clifford groups to orthogonal and orthogonal similitude groups (see the fundamental diagram (2.10)). Especially important for rank 6 forms is the relationship between a particular half-spin representation of the even Clifford group of a hyperbolic space, the second exterior power map, and the reduced pfaffian construction (see Proposition 3.6). Second, we perform explicit cocycle calculations in the Clifford group (see Theorem 2.25). These calculations require lifting similitudes to the Clifford group explicitly as well as computing products of such lifts.

## Motivation

The rest of this introduction will outline, as motivation, how $\mathscr{L}$-valued forms arise naturally in the theory of symmetric bilinear forms over schemes, and some situations in which the classical Hasse-Witt invariants have arithmetic interest.

### $\mathscr{L}$-valued forms

For a scheme $X$ with 2 invertible and $\mathscr{L}$ an invertible $\mathscr{O}_X$-module, the notion of a symmetric bilinear form over $X$ with values in $\mathscr{L}$ dates back to the early 1970s. Geyer/Harder/Knebusch/Scharlau [20] introduced the notion of a symmetric bilinear form over a global function field with values in the module of Kähler differentials. This notion enabled a consistent choice of local traces in order to generalize established residue theorems for forms over rational function fields. Mumford [37] introduced the notion of a locally free $\mathscr{O}_X$-module with a pairing into the sheaf of differentials, $\Omega^1_X$, to study theta characteristics on proper algebraic curves. A symmetric bilinear form $(\mathscr{E}, b, \mathscr{L})$ with values in $\mathscr{L}$ consists of a locally free $\mathscr{O}_X$-module $\mathscr{E}$ and a symmetric $\mathscr{O}_X$-bilinear morphism $b : \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{E} \to \mathscr{L}$. Such a form is called *nonsingular* if the associated adjoint morphism $\psi_b : \mathscr{E} \to \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{L})$ is an $\mathscr{O}_X$-module isomorphism.

The above two examples arise from trying to generalize the classical transfer (or trace) maps from the theory of forms over fields to the theory of forms over algebraic varieties. The general context in which transfer maps exist between Grothendieck-Witt groups of schemes has recently been established by the work of Gille [22], Nenashev [38], [39], and Calmès/Hornbostel [9], [10]. If $f : X \to Y$ is a proper morphism of connected, noetherian, regular $\operatorname{Spec} Z[\frac{1}{2}]$-schemes of relative dimension $d$, then the total derived direct image functor gives rise to a transfer map,

$$f_* : GW^{i+d}(X, \omega_f) \to GW^i(Y, \mathscr{O}_Y),$$

between the shifted derived (or coherent) Grothendieck-Witt groups introduced by Balmer [3], [4], [5], and Walter [52]. Here, $\omega_f$ is the relative dualizing sheaf and we use Grothendieck duality. In particular, in order to define the transfer along a proper morphism $f : X \to Y$, one is forced to consider $\omega_f$-valued bilinear forms on $X$.

## Hasse-Witt invariants

The $\mathscr{O}_X$-valued Grothendieck-Witt group is the algebraic analog of the $KO$-group of real vector bundles. Just as the classical Stiefel-Whitney invariants are important for studying real vector bundles on topological spaces, the analogous Hasse-Witt invariants are important for studying symmetric bilinear forms on schemes. The total Hasse-Witt invariant extends to a multiplicative map

$$GW^0(X, \mathscr{O}_X) \xrightarrow{w_i} H^*_{\text{ét}}(X, \boldsymbol{\mu}_2),$$

into the total mod 2 étale cohomology ring of $X$. The invariants help to classify symmetric bilinear forms. For example, if $X$ is the spectrum of a global field (not of characteristic 2), then symmetric bilinear forms are exactly classified up to isometry by their rank, signatures at archimedean places, discriminant (1st Hasse-Witt invariant), and 2nd Hasse-Witt invariant. The 1st and 2nd Hasse-Witt invariants also lift to invariants $e_1$ (the signed discriminant) and $e_2$ (the Clifford invariant) on the first and second fundamental filtration of the Witt group of a scheme. They are related to the Milnor conjecture over fields. Over arbitrary schemes, there is currently much activity around finding the right generalization of the Milnor conjecture.

In the 1980s, Serre [46] gave a formula relating the 2nd Hasse-Witt invariant of the trace form of a finite separable extension $K/k$, of fields of odd characteristic to a Galois theoretic characteristic class–the obstruction to an embedding problem in inverse Galois theory. For instance, this answers the question, "When can a Klein four Galois extension of fields be embedded in a quaternion extension?" Fröhlich [18] generalized this formula to arbitrary symmetric bilinear forms and Galois representations. Deligne [13] then related the Galois theoretic characteristic class (thus also the 2nd Hasse-Witt invariant, by Serre's formula) to the local root number of a Galois representation. The root number arises as part of the sign of the functional equation of the corresponding Artin $L$-function. In another direction, Esnault/Kahn/Viehweg [17] and Cassou-Noguès/Erez/Taylor [11], [12] generalized Serre's formula to finite tamely ramified coverings of schemes with odd ramification indices. Finally, Saito [44] has hinted at a version for proper morphisms of schemes, which extends to arbitrary orthogonal motives.

When $\mathscr{L}$ is not trivial, the corresponding Grothendieck-Witt groups are the algebraic analogues of twisted $KO$-groups. There is a substantial literature in physics applying twisted $KO$-theory to quantum field theory. In different guises, classes in twisted $KO$-theory can represent charges in boundary topological field theory and spaces of momenta of states in lattice models in solid state physics. For example, Kane/Mele [29] recently used an invariant in the twisted $KO$-theory of elliptic curves to define a new topological classification of the quantum spin Hall phase.

Topological analogues of the constructions in this dissertation could provide new cohomological invariants in twisted $KO$-theory and may in turn carry physical significance.

# Chapter 1

# Line bundle-valued symmetric bilinear forms

Throughout, let $X$ be a noetherian and separated scheme. For simplicity of exposition, we will assume that $X$ is connected. Invertible $\mathscr{O}_X$-modules (resp. locally free $\mathscr{O}_X$-modules of finite constant rank) will simply be referred to as *line bundles* (resp. *vector bundles*) on $X$. All unadorned tensor products and all internal hom sheaves between $\mathscr{O}_X$-modules will be over $\mathscr{O}_X$. We say that *2 is invertible on* $X$ or $\frac{1}{2} \in \mathscr{O}_X$ if multiplication by 2 on the structure sheaf has an inverse, or equivalently, if 2 is a unit in all local rings of points of $X$. We will eventually make this assumption. All sheaves, unless otherwise stated, will be considered in the étale topology. By convention, group schemes that come from base change from $\operatorname{Spec} \mathbb{Z}[\frac{1}{2}]$ (e.g. $\boldsymbol{\mu}_2$ and $\mathbb{G}_{\mathrm{m}}$) to $X$ will be denoted by the same symbol.

Concerning "sheafified" linear algebra, we wish to point out here that most of the work of this chapter is devoted to careful examination of exactly how familiar concepts from linear algebra and the theory of quadratic forms over fields and rings generalize to schemes. Almost every linear algebra construction applied to a vector bundle on a scheme is the sheaf associated to the correspondingly constructed presheaf. Accordingly, we will often define morphisms of sheaves associated to presheaves by defining a morphism of presheaves, and then implicitly consider the associated morphism of associated sheaves (sheafification is an exact functor).

## 1.1 Generalities on $\mathscr{L}$-valued bilinear forms

### 1.1.1 Definitions and notations

#### Bilinear forms

Let $X$ be a scheme and $\mathscr{L}$ a line bundle on $X$. An *($\mathscr{L}$-valued) bilinear form* on $X$ will mean a triple $(\mathscr{E}, b, \mathscr{L})$, where $\mathscr{E}$ is a vector bundle and $b : \mathscr{E} \times \mathscr{E} \to \mathscr{L}$ is an $\mathscr{O}_X$-bilinear morphism, equivalently, $b : \mathscr{E} \otimes \mathscr{E} \to \mathscr{L}$ is an $\mathscr{O}_X$-module morphism, equivalently, a choice of global section of $\mathscr{H}om(T^2\mathscr{E}, \mathscr{L})$. For sections $v, w \in \mathscr{E}(U)$ over $U \to X$, we will often write $b(v, w)$ in place of $b(U)(v \otimes w)$.

An ($\mathscr{L}$-valued) bilinear form on $X$ is called *symmetric* if $b : \mathscr{E} \otimes \mathscr{E} \to \mathscr{L}$ is invariant under the naive switch morphism $\mathscr{E} \otimes \mathscr{E} \to \mathscr{E} \otimes \mathscr{E}$ and is called *alternating* if $b$ vanishes when restricted to the diagonal tensors $\Delta : \mathscr{E} \to \mathscr{E} \otimes \mathscr{E}$. Equivalently $b$ is symmetric (resp. alternating) if it corresponds to a global section of $\mathscr{H}om(S^2\mathscr{E}, \mathscr{L})$ (resp. $\mathscr{H}om(\bigwedge^2\mathscr{E}, \mathscr{L})$) as a subsheaf of $\mathscr{H}om(T^2\mathscr{E}, \mathscr{L})$.

An *($\mathscr{L}$-valued) quadratic form* on $X$ will mean a triple $(\mathscr{E}, q, \mathscr{L})$, where $\mathscr{E}$ is a vector bundle and $q : \mathscr{E} \to \mathscr{L}$ is a map of sheaves satisfying the following two conditions:

- The following diagram of maps of sheaves is commutative,

$$\begin{tikzcd} \mathscr{O}_X \otimes \mathscr{E} \arrow[r, "\sim"] \arrow[d, "(-)^2 \otimes q"'] & \mathscr{E} \arrow[d, "q"] \\ \mathscr{O}_X \otimes \mathscr{L} \arrow[r, "\sim"] & \mathscr{L} \end{tikzcd}$$

  equivalently, on sections over $U \to X$, we have

$$q(av) = a^2 q(v), \quad \text{for all } a \in \mathscr{O}_X(U), v \in \mathscr{E}(U).$$

- The corresponding *polar form* $b_q : \mathscr{E} \otimes \mathscr{E} \to \mathscr{L}$, defined on sections over $U \to X$ by

$$b_q(v, w) = q(v + w) - q(v) - q(w), \quad \text{for all } v, w \in \mathscr{E}(U),$$

  is an $\mathscr{L}$-valued bilinear form on $X$.

Equivalently, $q$ is a choice of global section of the sheaf cokernel of the exact sequence

$$0 \to \mathscr{H}om(\textstyle\bigwedge^2\mathscr{E}, \mathscr{L}) \to \mathscr{H}om(T^2E, \mathscr{L}) \to \mathscr{Q}uad(\mathscr{E}, \mathscr{L}) \to 0. \tag{1.1}$$

This is the globalized version of the characterization of quadratic forms as equivalence classes of bilinear forms modulo alternating forms, see Knus [34, I §5.3.5] for further remarks on this characterization over rings.

Given an $\mathscr{L}$-valued bilinear form $b : \mathscr{E} \otimes \mathscr{E} \to \mathscr{L}$, the associated quadratic form

$$q_b : \mathscr{E} \xrightarrow{\Delta} \mathscr{E} \otimes \mathscr{E} \xrightarrow{b} \mathscr{L}, \tag{1.2}$$

is the image of $b$ under the map on global sections of exact sequence (1.1). Note that in general, not all quadratic forms are induced from global bilinear forms. The obstruction is the coboundary map to $H^1_{\text{ét}}(X, \mathscr{H}om(\bigwedge^2\mathscr{E}, \mathscr{L}))$, though this only causes a problem if 2 is not invertible.

**Lemma 1.1.** *Let $X$ be a scheme with $\frac{1}{2} \in \mathscr{O}_X$. Every quadratic form is the associated quadratic form of a (symmetric) bilinear form on $X$.*

*Proof.* This is a globalized version of the argument for rings. Via the various canonical isomorphisms, the dual of the defining quotient morphism $T^2(\mathscr{E}^\vee) \to \bigwedge^2(\mathscr{E}^\vee)$, gives rise to a morphism $\bigwedge^2\mathscr{E} \to T^2\mathscr{E}$. One checks that on sections over $U \to X$, this morphism is given by

$$v \wedge w \mapsto v \otimes w - w \otimes v.$$

Composing with $\frac{1}{2}$ yields a section of the defining quotient morphism $T^2\mathscr{E} \to \bigwedge^2\mathscr{E}$. Upon applying the exact functor $\mathscr{H}om(-, \mathscr{L})$, we find that the sequence (1.1) splits, and hence the cohomological coboundary maps vanish. Similarly, one may construct a morphism $\mathscr{H}om(S^2\mathscr{E}, \mathscr{L}) \to \mathscr{Q}uad(\mathscr{E}, \mathscr{L})$, which is invertible if 2 is invertible. $\square$

**Adjoint morphism**

An $\mathscr{L}$-valued bilinear form $b : \mathscr{E} \otimes \mathscr{E} \to \mathscr{L}$ has a corresponding *adjoint* morphism of $\mathscr{O}_X$-modules

$$\psi_b : \mathscr{E} \to \mathscr{H}om(\mathscr{E}, \mathscr{L}),$$

defined on sections over $U \to X$ by $\psi_b(v)(w) = b(v, w)$ for $v, w \in \mathscr{E}(U)$. A bilinear form $b$ is called *non-singular*, *regular*, or a *bilinear space* if its adjoint $\psi_b$ is an isomorphism of $\mathscr{O}_X$-modules. A quadratic form is called non-singular if its corresponding polar form is non-singular. Note that if 2 is not invertible on $X$ then every regular quadratic form has even rank.

On the category of coherent $\mathscr{O}_X$-modules, denote by $(-)^{\vee \mathscr{L}}$ the exact (contravariant) functor $\mathscr{H}om(-, \mathscr{L})$. Note that there is a canonical morphism of functors

$$\mathrm{can}^{\mathscr{L}} : \mathrm{id} \to ((-)^{\vee \mathscr{L}})^{\vee \mathscr{L}},$$

which is an isomorphism on the subcategory of locally free $\mathscr{O}_X$-modules. The category of vector bundles on $X$ together with $(-)^{\vee \mathscr{L}}$ and $\mathrm{can}^{\mathscr{L}}$ form an *exact category with duality* in the language of Balmer [3].

Define the *($\mathscr{L}$-valued) transpose* of a morphism $\psi : \mathscr{F} \to \mathscr{H}om(\mathscr{F}, \mathscr{L})$ of $\mathscr{O}_X$-modules by

$$\psi^t : \mathscr{F} \xrightarrow{\mathrm{can}^{\mathscr{L}}} \mathscr{H}om(\mathscr{H}om(\mathscr{F}, \mathscr{L}), \mathscr{L}) \xrightarrow{\psi^{\vee \mathscr{L}}} \mathscr{H}om(\mathscr{F}, \mathscr{L}). \tag{1.3}$$

Then a bilinear form $b$ is symmetric if and only if $\psi_b = \psi_b^t$. A bilinear form $b$ is called *skew-symmetric* if $\psi_b = -\psi_b^t$. Note that every alternating form is skew-symmetric, and conversely if $\frac{1}{2} \in \mathscr{O}_X$.

**Involutions on endomorphism algebras**

If $(\mathscr{E}, b, \mathscr{L})$ is a bilinear space, define the associated $\mathscr{O}_X$-algebra *anti-automorphism*

$$\sigma_b : \mathscr{E}nd(\mathscr{E}) \to \mathscr{E}nd(\mathscr{E})^{\mathrm{op}}$$

by the formula

$$\sigma_b(\varphi) = \psi_b|_U^{-1} \circ \varphi^{\vee \mathscr{L}|_U} \circ \psi_b|_U, \quad \text{for } \varphi : \mathscr{E}|_U \to \mathscr{E}|_U$$

on sections over $U \to X$. If $b$ is symmetric or alternating, then in fact $\sigma_b$ will be an $\mathscr{O}_X$-algebra *involution* (of the first kind), i.e. an $\mathscr{O}_X$-algebra anti-automorphism of order 2. In fact, the converse is true, see Knus/Parimala/Srinivas [33].

**Proposition 1.2.** *Let $X$ be a scheme and $\mathscr{E}$ a vector bundle on $X$. If $\sigma$ is an $\mathscr{O}_X$-algebra anti-automorphism of $\mathscr{E}nd(\mathscr{E})$ then there exists the structure of a bilinear space $(\mathscr{E}, b, \mathscr{L})$ on the vector bundle $\mathscr{E}$ such that $\sigma = \sigma_b$, moreover, the isomorphism class of $\mathscr{L}$ is uniquely determined and $b$ is determined up to multiplication by a global unit. If $\sigma$ is an involution on $\mathscr{E}nd(\mathscr{E})$ then the bilinear space $(\mathscr{E}, b, \mathscr{L})$ is either symmetric or alternating.*

Note however, that $\mathscr{E}nd(\mathscr{N} \otimes \mathscr{E}) \cong \mathscr{E}nd(\mathscr{E})$ as $\mathscr{O}_X$-algebras for any line bundle $\mathscr{N}$ on $X$ (see Remark 1.3 below). Thus an anti-automorphism $\sigma$ on $\mathscr{E}nd(\mathscr{E})$ that corresponds to a bilinear space $(\mathscr{E}, b, \mathscr{L})$ will also be an anti-automorphism on $\mathscr{E}nd(\mathscr{N} \otimes \mathscr{E})$ (via any choice of isomorphism of the endomorphism algebras), and so also corresponds to a bilinear space

$(\mathscr{N} \otimes \mathscr{E}, b', \mathscr{N}^{\otimes 2} \otimes \mathscr{L})$. Thus the isomorphism class of an endomorphism algebra with anti-automorphism $(\mathscr{E}nd(\mathscr{E}), \sigma)$ determines a bilinear space $(\mathscr{E}, b, \mathscr{L})$ only up to transformations of the form $(\mathscr{N} \otimes \mathscr{E}, n \otimes b, \mathscr{N}^{\otimes 2} \otimes \mathscr{L})$ where $(\mathscr{N}, n, \mathscr{N}^{\otimes 2})$ is a bilinear space of rank 1.

Later on, we will also consider Azumaya algebras with involution (of the first kind) on $X$. A locally free $\mathscr{O}_X$-algebra $\mathscr{A}$ of finite rank is an *Azumaya algebra* if the canonical $\mathscr{O}_X$-algebra homomorphism

$$\begin{array}{ccc} \mathscr{A} \otimes \mathscr{A}^{\mathrm{op}} & \to & \mathscr{E}nd(\mathscr{A}) \\ a \otimes b & \mapsto & c \mapsto acb \end{array}$$

is an isomorphism, where $\mathscr{A}^{\mathrm{op}}$ is the opposite algebra, and where $\mathscr{E}nd(\mathscr{A})$ is the endomorphism sheaf of $\mathscr{A}$ as an $\mathscr{O}_X$-module. For example, the endomorphism algebra $\mathscr{E}nd(\mathscr{E})$ of a vector bundle $\mathscr{E}$ on $X$ is an Azumaya algebra. In fact every Azumaya algebra is locally isomorphic in the étale topology to an endomorphism algebra, see Milne [36, IV Proposition 2.3]. An Azumaya algebra thus has rank $n^2$ for some $n$, which is called the *degree*. An anti-automorphism $\sigma$ on an Azumaya algebra $\mathscr{A}$, i.e. an $\mathscr{O}_X$-algebra isomorphism $\sigma : \mathscr{A} \to \mathscr{A}^{\mathrm{op}}$, is called an *involution* (of the first kind) if $\sigma^{\mathrm{op}} \circ \sigma = \mathrm{id}$. An Azumaya algebra with involution $(\mathscr{A}, \sigma)$ is locally isomorphic in the étale topology to an endomorphism algebra with involution, and thus by Proposition 1.2 corresponds to a bilinear space of either symmetric or alternating type. We say that an involution $\sigma$ on an Azumaya algebra is of *orthogonal type* or *symplectic type*, respectively.

**Orthogonal sum**

For bilinear forms $(\mathscr{E}, b, \mathscr{L})$ and $(\mathscr{E}', b', \mathscr{L})$, define the *orthogonal sum*

$$(\mathscr{E}, b, \mathscr{L}) \perp (\mathscr{E}', b', \mathscr{L}) = (\mathscr{E} \oplus \mathscr{E}', b + b', \mathscr{L})$$

on section over $U \to X$ by

$$(b + b')((v, v'), (w, w')) = b(v, w) + b'(v', w')$$

for $v, w \in \mathscr{E}(U)$ and $v', w' \in \mathscr{E}'(U)$. There's a similar notion of orthogonal sum for quadratic forms. The orthogonal sum of two symmetric forms is symmetric, two alternating forms is alternating, and two quadratic forms is quadratic.

**Tensor product**

For bilinear forms $(\mathscr{E}, b, \mathscr{L})$ and $(\mathscr{E}', b', \mathscr{L}')$, define the *tensor product*

$$(\mathscr{E}, b, \mathscr{L}) \otimes (\mathscr{E}', b', \mathscr{L}') = (\mathscr{E} \otimes \mathscr{E}', b \otimes b', \mathscr{L} \otimes \mathscr{L}')$$

on sections over $U \to X$ by

$$(b \otimes b')(v \otimes v', w \otimes w') = b(v, w) \otimes b'(v', w') \in \mathscr{L}(U) \otimes \mathscr{L}'(U)$$

for $v, w \in \mathscr{E}(U)$ and $v', w' \in \mathscr{E}'(U)$. There's a similar notion of tensor product for quadratic forms. The tensor product of two symmetric forms is symmetric, two alternating forms is symmetric, two quadratic forms is quadratic, a symmetric form with an alternating form is alternating, and a symmetric form with a quadratic form is quadratic.

### Diagonal forms

For global sections $a_1, \ldots, a_n$ of $\mathbb{G}_{\mathrm{m}}$ define the $\mathscr{L}$-valued symmetric bilinear space

$$\langle a_1, \ldots, a_n \rangle_{\mathscr{L}} = (\mathscr{L}^{\oplus n}, \langle a_1, \ldots, a_n \rangle_{\mathscr{L}}, \mathscr{L}^{\otimes 2})$$

by

$$\begin{array}{ccccc} \mathscr{L}^{\oplus n} & \otimes & \mathscr{L}^{\oplus n} & \to & \mathscr{L}^{\otimes 2} \\ (v_1, \ldots, v_n) & \otimes & (w_1, \ldots, w_n) & \mapsto & \sum_{i=1}^n a_i\, v_i \otimes w_i \end{array}$$

on sections over $U \to X$. It's the tensor product of the classical $\mathscr{O}_X$-valued symmetric bilinear space $\langle a_1, \ldots, a_n \rangle$ with the $\mathscr{L}^{\otimes 2}$-valued space $(\mathscr{L}, \langle 1 \rangle_{\mathscr{L}}, \mathscr{L}^{\otimes 2})$ of rank 1 on $X$.

### Isometries

An *isometry* of $\mathscr{L}$-valued bilinear forms $\varphi : (\mathscr{E}, b, \mathscr{L}) \xrightarrow{\sim} (\mathscr{E}', b', \mathscr{L})$ is an $\mathscr{O}_X$-module isomorphism $\varphi : \mathscr{E} \xrightarrow{\sim} \mathscr{E}'$ such that either of the following (equivalent) diagrams,

$$\begin{tikzcd} \mathscr{E} \otimes \mathscr{E} \arrow[r, "b"] \arrow[d, "\varphi \otimes \varphi"'] & \mathscr{L} \arrow[d, equal] \\ \mathscr{E}' \otimes \mathscr{E}' \arrow[r, "b'"] & \mathscr{L} \end{tikzcd} \qquad \begin{tikzcd} \mathscr{E} \arrow[r, "\psi_b"] \arrow[d, "\varphi"'] & \mathscr{H}om(\mathscr{E}, \mathscr{L}) \\ \mathscr{E}' \arrow[r, "\psi_{b'}"] & \mathscr{H}om(\mathscr{E}', \mathscr{L}) \arrow[u, "\varphi^{\vee \mathscr{L}}"'] \end{tikzcd} \tag{1.4}$$

of $\mathscr{O}_X$-modules commute. Note that the commutativity of the left-hand diagram (1.4) takes on the familiar formula,

$$b'(\varphi(v), \varphi(w)) = b(v, w), \quad \text{for all } v, w \in \mathscr{E}(U),$$

on sections over $U \to X$. An isometry of $\mathscr{L}$-valued quadratic forms $\varphi : (\mathscr{E}, q, \mathscr{L}) \xrightarrow{\sim} (\mathscr{E}', q', \mathscr{L})$ is an $\mathscr{O}_X$-module isomorphism $\varphi : \mathscr{E} \xrightarrow{\sim} \mathscr{E}'$ such that the following diagram,

$$\begin{tikzcd} \mathscr{E} \arrow[r, "q"] \arrow[d, "\varphi"'] & \mathscr{L} \arrow[d, equal] \\ \mathscr{E}' \arrow[r, "q'"] & \mathscr{L} \end{tikzcd} \tag{1.5}$$

of maps of $\mathscr{O}_X$-modules commutes. For bilinear or quadratic forms, we denote the group of isometries between $(\mathscr{E}, b, \mathscr{L})$ and $(\mathscr{E}', b', \mathscr{L})$ by $\mathrm{Isom}_X((\mathscr{E}, q, \mathscr{L}), (\mathscr{E}', q', \mathscr{L}))$.

### Similarity transformations

A *similarity (transformation)* or *similitude* between bilinear forms $(\mathscr{E}, b, \mathscr{L})$ and $(\mathscr{E}', b', \mathscr{L}')$ is a pair $(\varphi, \mu_\varphi)$ of $\mathscr{O}_X$-module isomorphisms $\varphi : \mathscr{E} \xrightarrow{\sim} \mathscr{E}'$ and $\mu_\varphi : \mathscr{L} \xrightarrow{\sim} \mathscr{L}'$ such that either of the following (equivalent) diagrams,

$$\begin{tikzcd} \mathscr{E} \otimes \mathscr{E} \arrow[r, "b"] \arrow[d, "\varphi \otimes \varphi"'] & \mathscr{L} \arrow[d, "\mu_\varphi"] \\ \mathscr{E}' \otimes \mathscr{E}' \arrow[r, "b'"] & \mathscr{L}' \end{tikzcd} \qquad \begin{tikzcd} \mathscr{E} \arrow[r, "\psi_b"] \arrow[d, "\varphi"'] & \mathscr{H}om(\mathscr{E}, \mathscr{L}) \\ \mathscr{E}' \arrow[r, "\psi_{b'}"] & \mathscr{H}om(\mathscr{E}', \mathscr{L}') \arrow[u, "\mu_\varphi^{-1} \varphi^{\vee \mathscr{L}}"'] \end{tikzcd} \tag{1.6}$$

of $\mathscr{O}_X$-modules commute, where

$$\begin{array}{rcl} \mu_\varphi^{-1}\varphi^{\vee\mathscr{L}} : \mathscr{H}om(\mathscr{E}',\mathscr{L}')(U) & \to & \mathscr{H}om(\mathscr{E},\mathscr{L})(U) \\ \psi & \mapsto & \mu_\varphi^{-1}|_U \circ \psi \circ \varphi|_U \end{array}$$

on sections over $U \to X$. Note that the commutativity of the left-hand diagram (1.6) takes on the familiar formula,

$$b'(\varphi(v), \varphi(w)) = \mu_\varphi(b(v,w)), \quad \text{for all } v, w \in \mathscr{E}(U)$$

on sections over $U \to X$. There's a similar notion of similarity transformation for quadratic forms. We denote the group of similarity transformations between $(\mathscr{E}, b, \mathscr{L})$ and $(\mathscr{E}', b', \mathscr{L}')$ by $\mathrm{Sim}_X((\mathscr{E}, b, \mathscr{L}), (\mathscr{E}', b', \mathscr{L}'))$.

### 1.1.2 The group schemes of isometries and similitudes

Let $(\mathscr{E}, q, \mathscr{L})$ be an $\mathscr{L}$-valued bilinear space on $X$. The presheaf of groups $\mathbf{Isom}(\mathscr{E}, b, \mathscr{L})$ given by

$$U \mapsto \mathrm{Isom}_U((\mathscr{E}|_U, b|_U, \mathscr{L}|_U), (\mathscr{E}|_U, b|_U, \mathscr{L}|_U))$$

is a sheaf for the étale topology on $X$ and is representable by a smooth group scheme on $X$ (see Demazure/Gabriel [14, III §5.2.3]) called the *group scheme of isometries* of $(\mathscr{E}, b, \mathscr{L})$. If $b$ is symmetric (resp. alternating), we also call $\mathbf{Isom}(\mathscr{E}, b, \mathscr{L})$ the *orthogonal group* $\mathbf{O}(\mathscr{E}, b, \mathscr{L})$ (resp. *symplectic group* $\mathbf{Sp}(\mathscr{E}, b, \mathscr{L})$).

The presheaf of groups $\mathbf{Sim}(\mathscr{E}, b, \mathscr{L})$ given by

$$U \mapsto \mathrm{Sim}_U((\mathscr{E}|_U, b|_U, \mathscr{L}|_U), (\mathscr{E}|_U, b|_U, \mathscr{L}|_U))$$

is a sheaf for the étale topology on $X$ and is representable by a smooth group scheme on $X$ called the *group scheme of similitudes* of $(\mathscr{E}, b, \mathscr{L})$. If $b$ is symmetric (resp. alternating), we also call $\mathbf{Sim}(\mathscr{E}, b, \mathscr{L})$ the *orthogonal similitude group* $\mathbf{GO}(\mathscr{E}, b, \mathscr{L})$ (resp. *symplectic similitude group* $\mathbf{GSp}(\mathscr{E}, b, \mathscr{L})$).

There's a natural (central) group scheme embedding $\mathbb{G}_\mathrm{m}(\mathscr{E}) \to \mathbf{Sim}(\mathscr{E}, b, \mathscr{L})$ given by homotheties, which defines the *projective similitude group scheme* as the sheaf quotient via the exact sequence

$$1 \to \mathbb{G}_\mathrm{m}(\mathscr{E}) \to \mathbf{Sim}(\mathscr{E}, b, \mathscr{L}) \to \mathbf{PSim}(\mathscr{E}, b, \mathscr{L}) \to 1.$$

of group schemes in the étale topology on $X$. If $b$ is symmetric (resp. alternating), we write $\mathbf{PGO}(\mathscr{E}, b, \mathscr{L})$ (resp. $\mathbf{PGSp}(\mathscr{E}, b, \mathscr{L})$) for this group scheme.

When $(\mathscr{E}, b, \mathscr{L})$ is fixed and no confusion will arise, we will simply write $\mathbf{Isom}$, $\mathbf{O}$, $\mathbf{Sp}$, $\mathbf{Sim}$, $\mathbf{GO}$, $\mathbf{GSp}$, $\mathbf{PSim}$, $\mathbf{PGO}$, and $\mathbf{PGSp}$ respectively, for the above groups. The group schemes of isometries and similitudes can be analogously defined for quadratic spaces.

*Remark* 1.3. If $\mathscr{E}$ is a vector bundle of rank $n$ on a scheme $X$, we will denote by $\mathbf{GL}(\mathscr{E})$ the general linear group scheme $\mathscr{E}$. The group schemes $\mathbf{GL}(\mathscr{E})$ and $\mathbf{GL}(\mathscr{E}')$ are isomorphic if and only if $\mathscr{E}' \cong \mathscr{E} \otimes \mathscr{L}$ for some line bundle $\mathscr{L}$. In particular, for any two line bundles $\mathscr{L}$ and $\mathscr{L}'$ on $X$, the group schemes $\mathbf{GL}(\mathscr{L})$ and $\mathbf{GL}(\mathscr{L}')$ are isomorphic. For any line bundle $\mathscr{L}$, there's a canonical isomorphism $\mathbb{G}_\mathrm{m} \xrightarrow{\sim} \mathbf{GL}(\mathscr{L})$, through which we shall identify $\mathbb{G}_\mathrm{m} = \mathbf{GL}(\mathscr{L})$. The only subtlety inherent in this identification arises when considering the pointed set of isomorphism classes of $\mathbf{GL}(\mathscr{L})$-torsors, for which the distinguished point is canonically the isomorphism class of $\mathscr{L}$. A similar remark should be made for identifications $\boldsymbol{\mu}_2 = \mathbf{O}(\mathscr{E}, b, \mathscr{L})$ where $(\mathscr{E}, b, \mathscr{L})$ is a bilinear space of rank 1. In what follows, we will keep track of these identifications, if not in our notation, then in our statements about torsors.

### 1.1.3 Torsor interpretations

Now assume that $\frac{1}{2} \in \mathscr{O}_X$ and that $X$ is endowed with the étale topology. For the abstract notion of (right) torsor for a group scheme over $X$ see Giraud [23], and for a down-to-earth summary in the case we're concerned with (including a translation into the language of stacks and gerbes), see Appendix A.

**Theorem 1.4.** *Let $X$ be a scheme with $\frac{1}{2} \in \mathscr{O}_X$ and endowed with the étale topology. Let $(\mathscr{E}, b, \mathscr{L})$ be a fixed $\mathscr{L}$-valued symmetric bilinear space of rank $n$ on $X$.*

*a) The category of $\mathbf{O}(\mathscr{E}, b, \mathscr{L})$-torsors is equivalent to the category of whose objects are $\mathscr{L}$-valued symmetric bilinear spaces of rank $n$ and whose morphisms are isometries.*

*b) The category of $\mathbf{GO}(\mathscr{E}, b, \mathscr{L})$-torsors is equivalent to the category whose objects are all symmetric bilinear spaces of rank $n$ with values in a line bundle and whose morphisms are similarity transformations.*

*c) The category of $\mathbf{PGO}(\mathscr{E}, b, \mathscr{L})$-torsors is equivalent to the category whose objects are Azumaya algebras of degree $n$ on $X$ with involution of orthogonal type and whose morphisms are involution preserving $\mathscr{O}_X$-algebra isomorphisms.*

*Proof.* We could not find an explicit proof of parts *a)* and *b* in the literature. For full details, see Theorem A.7. □

### 1.1.4 The multiplier sequence

The map assigning $(\varphi, \mu_\varphi) \mapsto \mu_\varphi$ on sections in the étale topology induces a group scheme homomorphism $\mu : \mathbf{Sim}(\mathscr{E}, b, \mathscr{L}) \to \mathbf{GL}(\mathscr{L})$. Identifying $\mathbf{GL}(\mathscr{L}) = \mathbb{G}_{\mathrm{m}}$, we define the *multiplier coefficient* $\mu : \mathbf{Sim}(\mathscr{E}, b, \mathscr{L}) \to \mathbb{G}_{\mathrm{m}}$. There's a canonical *multiplier sequence*.

**Proposition 1.5.** *For any scheme $X$ with $\frac{1}{2} \in \mathscr{O}_X$, the sequence of group schemes,*

$$1 \to \mathbf{Isom}(\mathscr{E}, b, \mathscr{L}) \to \mathbf{Sim}(\mathscr{E}, b, \mathscr{L}) \xrightarrow{\mu} \mathbb{G}_{\mathrm{m}} \to 1,$$

*is exact in the étale topology on $X$.*

*Proof.* The only non-obvious part is that $\mu$ is an epimorphism, but this follows from the fact that $\mu$ restricted to the central subgroup of homotheties is the squaring map and that the Kummer sequence in exact in the étale topology if 2 is invertible. □

*Remark* 1.6. The interpretation of the multiplier sequence on isomorphism classes of torsors is a follows. If $(\mathscr{E}, b, \mathscr{L})$ is a fixed $\mathscr{L}$-valued symmetric bilinear space of rank $n$ on $X$, then the map

$$H^1_{\text{ét}}(X, \mathbf{Isom}(\mathscr{E}, b, \mathscr{L})) \to H^1_{\text{ét}}(X, \mathbf{Sim}(\mathscr{E}, b, \mathscr{L}))$$

takes the isometry class of an $\mathscr{L}$-valued symmetric bilinear space of rank $n$ on $X$ to its similarity class. Under the identification $\mathbf{GL}(\mathscr{L}) = \mathbb{G}_{\mathrm{m}}$ the map

$$H^1_{\text{ét}}(X, \mathbf{Sim}(\mathscr{E}, b, \mathscr{L})) \to H^1_{\text{ét}}(X, \mathbb{G}_{\mathrm{m}}) \cong \mathrm{Pic}(X)$$

takes the similarity class of an $\mathscr{L}'$-valued symmetric bilinear space of rank $n$ on $X$ to the class of the line bundle $\mathscr{L}' \otimes \mathscr{L}^{\vee}$ in $\mathrm{Pic}(X)$.

In the previous proof, we implicitly considered a canonical commutative diagram with exact rows and columns,

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
1 \arrow[r] & \mu_2 \arrow[r] \arrow[d] & \mathbb{G}_{\mathrm{m}} \arrow[r, "2"] \arrow[d] & \mathbb{G}_{\mathrm{m}} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathbf{Isom} \arrow[r] \arrow[d] & \mathbf{Sim} \arrow[r, "\mu"] \arrow[d] & \mathbb{G}_{\mathrm{m}} \arrow[r] & 1 \\
 & \mathbf{PIsom} \arrow[r] \arrow[d] & \mathbf{PSim} \arrow[d] & & \\
 & 1 & 1 & &
\end{tikzcd}$$

of group schemes in the étale topology on $X$, where the quotient group scheme $\mathbf{PIsom}(\mathscr{E}, b, \mathscr{L})$ is called the *projective isometry group* of the bilinear form $(\mathscr{E}, b, \mathscr{L})$. As usual, if $b$ is symmetric (resp. alternating), we write $\mathbf{PO}(\mathscr{E}, q, \mathscr{L})$ (resp. $\mathbf{PSp}(\mathscr{E}, q, \mathscr{L})$). From the above diagram, there's an induced isomorphism of group schemes,

$$\mathbf{PIsom}(\mathscr{E}, b, \mathscr{L}) \xrightarrow{\sim} \mathbf{PSim}(\mathscr{E}, b, \mathscr{L}),$$

in the étale topology on $X$ (if 2 is invertible).

## 1.2 Proper similarity transformations

From now on we will assume that $\frac{1}{2} \in \mathscr{O}_X$ and that $X$ is endowed with the étale topology. In particular, by Lemma 1.1, the notions of symmetric bilinear form and quadratic form on $X$ are equivalent. Many of our results have analogies if 2 is not invertible on $X$, but require moving to the flat site and making slightly different constructions. This should be a project for future work.

### 1.2.1 The discriminant form

#### The determinant

Let $(\mathscr{E}, b, \mathscr{L})$ be an $\mathscr{L}$-valued bilinear space on of rank $n$ on $X$. Applying the determinant functor to the adjoint morphism

$$\psi_b : \mathscr{E} \xrightarrow{\sim} \mathscr{H}om(\mathscr{E}, \mathscr{L}),$$

yields an $\mathscr{O}_X$-module morphism

$$\det \mathscr{E} \xrightarrow{\det \psi_b} \det \mathscr{H}om(\mathscr{E}, \mathscr{L}) \xrightarrow{\text{can}} \mathscr{H}om(\det \mathscr{E}, \mathscr{L}^{\otimes n}),$$

of line bundles on $X$. Here can : $\det \mathscr{H}om(\mathscr{E}, \mathscr{L}) \xrightarrow{\sim} \mathscr{H}om(\det \mathscr{E}, \mathscr{L}^{\otimes n})$ is the canonical isomorphism of line bundles given by

$$f_1 \wedge \cdots \wedge f_n \mapsto (v_1 \wedge \cdots \wedge v_n \mapsto \det(f_i(v_j))_{ij})$$

on sections over $U \to X$. We will write $\det(\mathscr{E}, b, \mathscr{L}) = (\det \mathscr{E}, \det b, \mathscr{L}^{\otimes n})$ for the $\mathscr{L}^{\otimes n}$-valued bilinear form of rank 1 on $X$ whose adjoint morphism is the above canonical composition $\text{can} \circ \det \psi_b$. We call $\det(\mathscr{E}, b, \mathscr{L})$ the *determinant form* of $(\mathscr{E}, b, \mathscr{L})$. Then $\det(\mathscr{E}, b, \mathscr{L})$ is regular if $(\mathscr{E}, b, \mathscr{L})$ is regular. Note that under the above identifications, $\det b$ is given by

$$\begin{array}{ccc} \det \mathscr{E} \otimes \det \mathscr{E} & \xrightarrow{\det b} & \mathscr{L}^{\otimes n} \\ v_1 \wedge \cdots \wedge v_n \otimes w_1 \wedge \cdots \wedge w_n & \longmapsto & \det(b(v_i, w_j))_{ij} \end{array}$$

on section over $U \to X$.

The determinant form is functorial with respect to similarity transformations. Indeed, if $(\varphi, \mu_\varphi) : (\mathscr{E}, b, \mathscr{L}) \to (\mathscr{E}', b', \mathscr{L}')$ is a similarity transformation, then there's an induced similarity transformation $(\det \varphi, \mu_\varphi^{\otimes n}) : (\det \mathscr{E}, \det b, \mathscr{L}^{\otimes n}) \to (\det \mathscr{E}', \det b', \mathscr{L}'^{\otimes n})$. The induced homomorphism of group schemes $\det : \mathbf{Sim}(\mathscr{E}, b, \mathscr{L}) \to \mathbf{Sim}(\det \mathscr{E}, \det b, \mathscr{L}^{\otimes n})$ factors through the forgetful embedding into the general linear group,

$$\begin{tikzcd} \mathbf{Sim}(\mathscr{E}, b, \mathscr{L}) \arrow[r] \arrow[d, "\det"'] & \mathbf{GL}(\mathscr{E}) \arrow[d, "\det"] \\ \mathbf{Sim}(\det \mathscr{E}, \det b, \mathscr{L}^{\otimes n}) \arrow[r, "\sim"] & \mathbf{GL}(\det \mathscr{E}) \end{tikzcd}$$

Identifying $\mathbf{Sim}(\det \mathscr{E}, \det b, \mathscr{L}^{\otimes n}) = \mathbf{GL}(\det \mathscr{E}) = \mathbb{G}_{\mathrm{m}}$, we define the *determinant homomorphism*

$$\det : \mathbf{Sim}(\mathscr{E}, b, \mathscr{L}) \to \mathbb{G}_{\mathrm{m}}.$$

**The discriminant**

For forms of even rank, the classical discriminant form generalizes to bilinear forms with values in line bundles. The analogue of the signed discriminant for $\mathscr{L}$-valued forms appears in Parimala/Sridharan [40, §4], where it was used to construct the discriminant form of a general Azumaya algebra with involution over an arbitrary scheme.

**Definition 1.7.** Let $(\mathscr{E}, b, \mathscr{L})$ be an $\mathscr{L}$-valued bilinear space of even rank $n = 2m$ on $X$. Define the *discriminant form* of $(\mathscr{E}, b, \mathscr{L})$ as the $\mathscr{O}_X$-valued bilinear space $\operatorname{disc}(\mathscr{E}, b, \mathscr{L}) = (\mathscr{L}^{\vee \otimes m} \otimes \det \mathscr{E}, \operatorname{disc} b, \mathscr{O}_X)$ of rank 1 given by the composition of the tensor product of the determinant form of with $\langle 1 \rangle_{\mathscr{L}^{\otimes m}}$

$$\operatorname{disc} b : (\mathscr{L}^{\vee \otimes m} \otimes \det \mathscr{E}) \otimes (\mathscr{L}^{\vee \otimes m} \otimes \det \mathscr{E}) \xrightarrow{\langle 1 \rangle_{\mathscr{L}^{\vee \otimes m}} \otimes \det b} \mathscr{L}^{\vee \otimes n} \otimes \mathscr{L}^{\otimes n} \xrightarrow{\text{ev}} \mathscr{O}_X,$$

with the canonical evaluation pairing.

The discriminant form has a finer functorial behavior with respect to similarity transformations than does the determinant form.

**Proposition 1.8.** *Let* $(\varphi, \mu_\varphi) : (\mathscr{E}, b, \mathscr{L}) \to (\mathscr{E}', b', \mathscr{L}')$ *be a similarity transformation of bilinear spaces of even rank on* $X$*. Then there's an induced isometry* $\mathrm{disc}(\varphi, \mu_\varphi) : \mathrm{disc}(\mathscr{E}, b, \mathscr{L}) \to \mathrm{disc}(\mathscr{E}', b', \mathscr{L}')$*, of discriminant forms.*

*Proof.* For a similarity $(\varphi, \mu_\varphi) : (\mathscr{E}, b, \mathscr{L}) \to (\mathscr{E}', b', \mathscr{L}')$ of bilinear spaces of even rank $n = 2m$, define

$$\mathrm{disc}(\varphi, \mu_\varphi) = (\mu_\varphi^{-1\vee})^{\otimes m} \otimes \det \varphi : \mathscr{L}^{\vee \otimes m} \otimes \det \mathscr{E} \to \mathscr{L}'^{\vee \otimes m} \otimes \det \mathscr{E}'$$

and then using a routine diagram chase on sections, check that the following diagram,

$$\begin{tikzcd}
\mathscr{L}^{\vee\otimes m} \otimes \det \mathscr{E} \otimes \mathscr{L}^{\vee\otimes m} \otimes \det \mathscr{E} \arrow[r, "\langle 1 \rangle_{\mathscr{L}^{\vee\otimes m}} \otimes \det b"] \arrow[d, "(\mathrm{disc}(\varphi, \mu_\varphi))^{\otimes 2}"'] & \mathscr{L}^{\vee\otimes n} \otimes \mathscr{L}^{\otimes n} \arrow[r, "\mathrm{ev}"] \arrow[d, "\mu_\varphi^{-1\vee\otimes n} \otimes \mu_\varphi^{\otimes n}"] & \mathscr{O}_X \arrow[d, equal] \\
\mathscr{L}'^{\vee\otimes m} \otimes \det \mathscr{E}' \otimes \mathscr{L}'^{\vee\otimes m} \otimes \det \mathscr{E}' \arrow[r, "\langle 1 \rangle_{\mathscr{L}'^{\vee\otimes m}} \otimes \det b'"] & \mathscr{L}'^{\vee\otimes n} \otimes \mathscr{L}'^{\otimes n} \arrow[r, "\mathrm{ev}"] & \mathscr{O}_X
\end{tikzcd}$$

of $\mathscr{O}_X$-module morphisms is commutative. □

**Proposition 1.9.** *Let* $(\mathscr{E}, b, \mathscr{L})$ *be a bilinear space of rank* $n$ *(for any* $n$*) on* $X$*. Then the group scheme homomorphisms* $\det$ *and* $\mu : \mathbf{Sim}(\mathscr{E}, b, \mathscr{L}) \to \mathbb{G}_{\mathrm{m}}$*, are related by*

$$\det{}^2 = \mu^n.$$

*Proof.* Note that in the case of $n$ even, this is a corollary of Proposition 1.8. We give a different argument that works in all cases. For a section $(\varphi, \mu_\varphi)$ of $\mathbf{Sim}(\mathscr{E}, b, \mathscr{L})$ over $U \to X$, we have canonically identified sections $\det(\varphi), \mu_\varphi \in \mathbb{G}_{\mathrm{m}}(U)$. By the commutativity of right-hand diagram (1.6), we have

$$\big(\mu_\varphi^{-1} \varphi^{\vee_{\mathscr{L}}|_U}\big) \circ (a_b|_U \circ \varphi \circ a_b|_U^{-1}) = \mathrm{id}_U : \mathscr{H}om(\mathscr{E}, \mathscr{L})(U) \to \mathscr{H}om(\mathscr{E}, \mathscr{L})(U).$$

Note that as sections of $\mathbb{G}_{\mathrm{m}}$, we have the equalities

$$\det\big(\mu_\varphi^{-1} \varphi^{\vee_{\mathscr{L}}|_U}\big) = \det(\varphi)\, (\mu_\varphi^{-1})^n, \quad \det(a_b|_U \circ \varphi \circ a_b{}^{-1}|_U) = \det(\varphi)$$

and hence from above we see that $\det(\varphi)^2 = \mu_\varphi^n \in \mathbb{G}_{\mathrm{m}}(U)$. □

### 1.2.2 Proper similarity transformations

For bilinear spaces of even rank, similarity transformations that induce, via Proposition 1.9, the identity on the discriminant form are called *proper*.

### Alternating case

The existence of the pfaffian makes the discriminant form uninteresting for alternating spaces and all similarity transformations proper, see Knus [34, §9.5.4]. We say that a discriminant form is trivial if it's isometric to the standard multiplication isomorphism $\langle 1\rangle : \mathscr{O}_X \otimes \mathscr{O}_X \xrightarrow{\sim} \mathscr{O}_X$.

**Proposition 1.10.** *For any alternating space $(\mathscr{E}, b, \mathscr{L})$ of (necessarily even) rank $n$ on $X$, the discriminant form is trivial and any similarity transformation $(\varphi, \mu_\varphi) : (\mathscr{E}, b, \mathscr{L}) \to (\mathscr{E}, b, \mathscr{L})$ induces the identity on the discriminant form.*

### Symmetric case

For $\mathscr{L}$-valued symmetric bilinear spaces $(\mathscr{E}, b, \mathscr{L})$ of even rank $n = 2m$ on $X$, the sheaf morphism $\mathbf{GO}(\mathscr{E}, b, \mathscr{L}) \to \mathbf{O}(\mathrm{disc}(\mathscr{E}, b, \mathscr{L}))$ induced by Proposition 1.8, defines the *similitude discriminant homomorphism*,

$$\mathrm{disc} : \mathbf{GO}(\mathscr{E}, b, \mathscr{L}) \to \boldsymbol{\mu}_2,$$

via the identification $\mathbf{O}(\mathrm{disc}(\mathscr{E}, b, \mathscr{L})) = \boldsymbol{\mu}_2$. The determinant, discriminant, and multiplier homomorphisms for the similitude groups are related by the formula,

$$\det = \mathrm{disc} \cdot \mu^m$$

under the multiplication homomorphism $\boldsymbol{\mu}_2 \times \mathbb{G}_\mathrm{m} \to \mathbb{G}_\mathrm{m}$. This justifies the formula,

$$\mathrm{disc} = \frac{\det}{\mu^m},$$

that is often found in the literature.

**Definition 1.11.** Let $(\mathscr{E}, b, \mathscr{L})$ be an $\mathscr{L}$-valued symmetric bilinear space of even rank $n$ on $X$.

*a)* Define the group scheme $\mathbf{GSO}(\mathscr{E}, b, \mathscr{L})$ of *proper orthogonal similitudes* as the sheaf kernel of the similitude discriminant homomorphism $\mathrm{disc} : \mathbf{GO}(\mathscr{E}, b, \mathscr{L}) \to \boldsymbol{\mu}_2$.

*b)* Define the *special orthogonal* (or the *proper orthogonal*) group scheme $\mathbf{SO}(\mathscr{E}, b, \mathscr{L})$ as the intersection of $\mathbf{GSO}(\mathscr{E}, b, \mathscr{L})$ and $\mathbf{O}(\mathscr{E}, b, \mathscr{L})$ inside $\mathbf{GO}(\mathscr{E}, b, \mathscr{L})$.

*Remark* 1.12. The determinant form is also functorial with respect to isometries. Indeed, if $\varphi : (\mathscr{E}, b, \mathscr{L}) \to (\mathscr{E}', b', \mathscr{L})$ is an isometry of $\mathscr{L}$-valued bilinear space (of any rank) on $X$, then there's an induced isometry $\det \varphi : \det(\mathscr{E}, b, \mathscr{L}) \to \det(\mathscr{E}', b', \mathscr{L})$ of determinant forms. The induced group scheme homomorphism $\det : \mathbf{O}(\mathscr{E}, b, \mathscr{L}) \to \mathbf{O}(\det(\mathscr{E}, b, \mathscr{L}))$ defines the *determinant homomorphism* for the orthogonal group $\det : \mathbf{O}(\mathscr{E}, b, \mathscr{L}) \to \boldsymbol{\mu}_2$ via the identification $\mathbf{O}(\det(\mathscr{E}, b, \mathscr{L})) = \boldsymbol{\mu}_2$. The determinant homomorphism for the orthogonal group fits together with the determinant and discriminant (in the case of even rank) homomorphisms for the similitude group in commutative diagrams,

$$\begin{tikzcd} \mathbf{O}(\mathscr{E}, b, \mathscr{L}) \arrow[r, "\det"] \arrow[d] & \boldsymbol{\mu}_2 \arrow[d] \\ \mathbf{GO}(\mathscr{E}, b, \mathscr{L}) \arrow[r, "\det"] & \mathbb{G}_\mathrm{m} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{O}(\mathscr{E}, b, \mathscr{L}) \arrow[r, "\det"] \arrow[d] & \boldsymbol{\mu}_2 \arrow[d, equal] \\ \mathbf{GO}(\mathscr{E}, b, \mathscr{L}) \arrow[r, "\mathrm{disc}"] & \boldsymbol{\mu}_2 \end{tikzcd}$$

of group schemes. It is for this reason that we make the identification $\det = \mathrm{disc}|_{\mathbf{O}(\mathscr{E},b,\mathscr{L})}$ of the determinant homomorphism for the orthogonal group and the restriction (which we also call $\mathrm{disc}$) of the discriminant homomorphism to the orthogonal group of an even rank form, though they are not canonically equal. In particular, in the case of even rank, the special orthogonal group $\mathbf{SO}(\mathscr{E},b,\mathscr{L})$ is also the kernel of the determinant homomorphism for the orthogonal group.

**Proposition 1.13.** *Let $(\mathscr{E},b,\mathscr{L})$ be an $\mathscr{L}$-valued symmetric bilinear space of rank $n=2m$ on $X$.*

*a) There's an exact sequence,*

$$1 \to \mathbf{SO}(\mathscr{E},b,\mathscr{L}) \to \mathbf{O}(\mathscr{E},b,\mathscr{L}) \xrightarrow{\det} \boldsymbol{\mu}_2 \to 1, \tag{1.7}$$

*of group schemes in the étale topology on $X$, called the* determinant sequence *for the orthogonal group.*

*b) There are exact sequences,*

$$1 \to \mathbf{SO}(\mathscr{E},b,\mathscr{L}) \to \mathbf{GSO}(\mathscr{E},b,\mathscr{L}) \xrightarrow{\mu} \mathbb{G}_{\mathrm{m}} \to 1,$$

*(called the* proper multiplier sequence*), and*

$$1 \to \mathbf{GSO}(\mathscr{E},b,\mathscr{L}) \to \mathbf{GO}(\mathscr{E},b,\mathscr{L}) \xrightarrow{\mathrm{disc}} \boldsymbol{\mu}_2 \to 1,$$

*(called the* discriminant sequence *for the similitude group) and a commutative diagram with exact rows and columns,*

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
1 \arrow[r] & \mathbf{SO} \arrow[r] \arrow[d] & \mathbf{GSO} \arrow[r, "\mu"] \arrow[d] & \mathbb{G}_{\mathrm{m}} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathbf{O} \arrow[r] \arrow[d, "\det"'] & \mathbf{GO} \arrow[r, "\mu"] \arrow[d, "\mathrm{disc}"'] & \mathbb{G}_{\mathrm{m}} \arrow[r] & 1 \\
 & \boldsymbol{\mu}_2 \arrow[r, equal] \arrow[d] & \boldsymbol{\mu}_2 \arrow[d] & & \\
 & 1 & 1 & &
\end{tikzcd}$$

*of group schemes in the étale topology on $X$.*

*Proof.* To show that $\det$ (resp. $\mathrm{disc}$) is an epimorphism, it's sufficient to find an improper isometry (resp. similitude), which is standard. To show that $\mu$ is an epimorphism in the proper multiplier sequence, we follow the proof of Proposition 1.5 noting that homotheties are proper similitudes. □

### 1.2.3 Proper torsor interpretations

**Theorem 1.14.** *Let $X$ be a scheme with $\frac{1}{2} \in \mathscr{O}_X$ and endowed with the étale topology. Let $(\mathscr{E}, b, \mathscr{L})$ be a fixed $\mathscr{L}$-valued symmetric bilinear form of rank $n$ on $X$.*

*a) The category of $\mathbf{SO}(\mathscr{E}, b, \mathscr{L})$-torsors is equivalent to the category whose objects are pairs $((\mathscr{E}', b', \mathscr{L}), \psi')$ consisting of an $\mathscr{L}$-valued symmetric bilinear space of rank $n$ together with an isometry $\psi' : \operatorname{disc}(\mathscr{E}, b, \mathscr{L}) \to \operatorname{disc}(\mathscr{E}', b', \mathscr{L})$ of discriminant forms, and whose morphisms between objects $((\mathscr{E}', b', \mathscr{L}), \psi')$ and $((\mathscr{E}'', b'', \mathscr{L}), \psi'')$ are isometries $\varphi : (\mathscr{E}', b', \mathscr{L}) \to (\mathscr{E}'', b'', \mathscr{L})$ such that $\psi'' = \operatorname{disc}(\varphi) \circ \psi'$.*

*b) Let $n$ be even. The category of $\mathbf{GSO}(\mathscr{E}, b, \mathscr{L})$-torsors is equivalent to the category whose objects are pairs $((\mathscr{E}', b', \mathscr{L}'), \psi')$ consisting of an $\mathscr{L}'$-valued symmetric bilinear space of rank $n$ (for some line bundle $\mathscr{L}'$ on $X$) together with an isometry $\psi' : \operatorname{disc}(\mathscr{E}, b, \mathscr{L}) \to \operatorname{disc}(\mathscr{E}', b', \mathscr{L}')$ of discriminant forms, and whose morphisms between any two objects $((\mathscr{E}', b', \mathscr{L}'), \psi')$ and $((\mathscr{E}'', b'', \mathscr{L}''), \psi'')$ are similarities $\varphi : (\mathscr{E}', b', \mathscr{L}') \to (\mathscr{E}'', b'', \mathscr{L}'')$ such that $\psi'' = \operatorname{disc}(\varphi) \circ \psi'$.*

*Proof.* We could not find an explicit proof of this assertion in the literature. For full details, see Theorem A.7. □

## 1.3 Bilinear forms of odd rank

We assume that $\frac{1}{2} \in \mathscr{O}_X$ and that $X$ is endowed with the étale topology. The study of bilinear forms of odd rank reduces to the study of $\mathscr{O}_X$-valued bilinear forms.

**Theorem 1.15.** *If $\mathscr{L}$ is not a square in the Picard group $\operatorname{Pic}(X)$ of $X$, then any $\mathscr{L}$-valued bilinear space has even rank.*

*Proof.* Let $(\mathscr{E}, b, \mathscr{L})$ be a bilinear space of rank $n$. Comparing determinants (via the polar form $\psi_b$) yields an isomorphism $\det(E)^{\otimes 2} \cong \mathscr{L}^{\otimes n}$ of line bundles. Thus we see that either $r$ must be even or $\mathscr{L}$ is a square in $\operatorname{Pic}(X)$ (up to an $r$-torsion element, which is itself a square). An alternate proof can be found in [7, Theorem 3.7], also see Proposition 1.38. □

Thus every bilinear space of odd rank has values in the square of some line bundle. As we shall see, the simple structure of the odd rank similitude orthogonal group allows for a canonical choice of square root.

### 1.3.1 The similitude group in odd rank

While there is no similitude discriminant for bilinear forms of odd rank, the simple structure of the similitude group scheme in odd rank provides an analogue.

#### The similitude absolute value

The following construction is quite classical, see Dieudonné [15, II §13], yet there is no standard name for it in the literature.

**Definition 1.16.** Let $(\mathscr{E}, b, \mathscr{L})$ be an $\mathscr{L}$-valued bilinear space of odd rank $n = 2m + 1$ on $X$. Define the *absolute value form* of $(\mathscr{E}, b, \mathscr{L})$ as the $\mathscr{L}$-valued (symmetric) bilinear space $|\mathscr{E}, b, \mathscr{L}| = (|\mathscr{E}|, |b|, \mathscr{L})$ of rank 1, where $|\mathscr{E}| = \mathscr{L}^{\vee \otimes m} \otimes \det \mathscr{E}$ and $|b|$ is given by composition of the tensor product of the determinant form with $\langle 1 \rangle_{\mathscr{L}^{\vee \otimes m}}$,

$$|b| : (\mathscr{L}^{\vee \otimes m} \otimes \det \mathscr{E}) \otimes (\mathscr{L}^{\vee \otimes m} \otimes \det \mathscr{E}) \xrightarrow{\mathrm{id} \otimes \det b} \mathscr{L}^{\vee \otimes 2m} \otimes \mathscr{L}^{\otimes n} \xrightarrow{\mathrm{ev}} \mathscr{L},$$

with the canonical evaluation pairing.

The absolute value form is functorial with respect to similarity transformations. Indeed, if $(\varphi, \mu_\varphi) : (\mathscr{E}, b, \mathscr{L}) \to (\mathscr{E}', b', \mathscr{L}')$ is a similarity, then there's an induced similarity $(\det \varphi, \mu_\varphi) : |\mathscr{E}, b, \mathscr{L}| \to |\mathscr{E}', b', \mathscr{L}'|$. The induced homomorphism of group schemes $|\cdot| : \mathbf{Sim}(\mathscr{E}, b, \mathscr{L}) \to \mathbf{Sim}(|\mathscr{E}, b, \mathscr{L}|)$ defines the *absolute value homomorphism* for the similitude group

$$|\cdot| : \mathbf{Sim}(\mathscr{E}, b, \mathscr{L}) \to \mathbb{G}_{\mathrm{m}}$$

via the identification $\mathbf{Sim}(|\mathscr{E}, b, \mathscr{L}|) = \mathbb{G}_{\mathrm{m}}$.

**Proposition 1.17.** *Let $(\mathscr{E}, b, \mathscr{L})$ be an $\mathscr{L}$-valued bilinear space of odd rank $n = 2m + 1$ on $X$. Then the group scheme homomorphisms* $\det$ *and* $\mu$ *and* $|\cdot| : \mathbf{Sim}(\mathscr{E}, b, \mathscr{L}) \to \mathbb{G}_{\mathrm{m}}$, *are related by*

$$|\cdot|^2 = \mu, \quad \textit{and} \quad |\cdot|^n = \det. \tag{1.8}$$

*Proof.* See the statement and proof of Proposition 1.9. □

### The normalized form

**Definition 1.18.** Let $(\mathscr{E}, b, \mathscr{L})$ be an $\mathscr{L}$-valued bilinear space of odd rank $n = 2m + 1$ on $X$. Define the *normalized form* of $(\mathscr{E}, b, \mathscr{L})$ as the $\mathscr{O}_X$-valued bilinear space $u(\mathscr{E}, b, \mathscr{L}) = (|\mathscr{E}|^\vee \otimes \mathscr{E}, u(b), \mathscr{O}_X)$ of rank 1 given by composition of the tensor product with the dual of the absolute value,

$$u(b) : (|\mathscr{E}|^\vee \otimes \mathscr{E}) \otimes (|\mathscr{E}|^\vee \otimes \mathscr{E}) \xrightarrow{|b|^{-1\vee} \otimes b} \mathscr{L}^\vee \otimes \mathscr{L} \xrightarrow{\mathrm{ev}} \mathscr{O}_X,$$

with the canonical evaluation pairing.

The normalized form has good functorial behavior with respect to similarity transformations.

**Proposition 1.19.** *Let $(\varphi, \mu_\varphi) : (\mathscr{E}, b, \mathscr{L}) \to (\mathscr{E}', b', \mathscr{L}')$ be a similarity transformation of bilinear spaces of odd rank on $X$. Then there's an induced isometry $u(\varphi, \mu_\varphi) = \det \varphi^{-1\vee} \otimes \varphi : u(\mathscr{E}, b, \mathscr{L}) \to n(\mathscr{E}', b', \mathscr{L}')$ of normalized forms. In particular, the isometry class of the normalized form solely depends on the similarity class of the form.*

*Proof.* This is a straightforward modification of the proof of Proposition 1.8. □

**Lemma 1.20.** *Let $(\mathscr{E}, b, \mathscr{L})$ be a bilinear space of odd rank $n = 2m + 1$ on $X$. Then the discriminant* $\mathrm{disc}(u(\mathscr{E}, b, \mathscr{L}))$ *of the normalized form is isometric to the trivial discriminant.*

*Proof.* This is an easy exercise in unraveling the definitions. □

### Structure of odd rank orthogonal similitude groups

We will now work exclusively with symmetric bilinear spaces of odd rank on $X$. As compared to the case of even rank, we can only define the special orthogonal group by means of the determinant homomorphism, compare with Remark 1.12.

**Definition 1.21.** Let $(\mathscr{E}, b, \mathscr{L})$ be a symmetric bilinear space of odd rank on $X$. Define the *special orthogonal* group scheme $\mathbf{SO}(\mathscr{E}, b, \mathscr{L})$ as the kernel of the determinant homomorphism $\det : \mathbf{O}(\mathscr{E}, b, \mathscr{L}) \to \boldsymbol{\mu}_2$ for the orthogonal group.

**Lemma 1.22.** *The group scheme homomorphism induced from Proposition 1.19, called the* normalization homomorphism*, has imagine in the special orthogonal group scheme*

$$u : \mathbf{GO}(\mathscr{E}, b, \mathscr{L}) \to \mathbf{SO}(u(\mathscr{E}, b, \mathscr{L})).$$

*Proof.* By Proposition 1.19, $u$ has image in $\mathbf{O}(u(\mathscr{E}, b, \mathscr{L}))$. To show that it has image in the special orthogonal group scheme, we appeal to the right-hand part of formula (1.8). □

The special orthogonal group in odd rank is also the kernel of the absolute value homomorphism for the similitude group.

**Proposition 1.23.** *Let $(\mathscr{E}, b, \mathscr{L})$ be an $\mathscr{L}$-valued symmetric bilinear space of odd rank on $X$.*

*a) There's an exact sequence,*

$$1 \to \mathbf{SO}(\mathscr{E}, b, \mathscr{L}) \to \mathbf{O}(\mathscr{E}, b, \mathscr{L}) \xrightarrow{\det} \boldsymbol{\mu}_2 \to 1, \tag{1.9}$$

*of group schemes in the étale topology on $X$, called the* determinant sequence *for the orthogonal group.*

*b) There's an exact sequence,*

$$1 \to \mathbf{SO}(\mathscr{E}, b, \mathscr{L}) \to \mathbf{GO}(\mathscr{E}, b, \mathscr{L}) \xrightarrow{|\cdot|} \mathbb{G}_{\mathrm{m}} \to 1,$$

*of group schemes in the étale topology on $X$. This sequence is split by sending a section of $\mathbb{G}_{\mathrm{m}}$ to the corresponding homothety. There's a commutative diagram with exact rows and columns,*

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
 & \mathbf{SO} \arrow[r, equal] \arrow[d] & \mathbf{SO} \arrow[d] & & \\
1 \arrow[r] & \mathbf{O} \arrow[r] \arrow[d, "\det"'] & \mathbf{GO} \arrow[r, "\mu"] \arrow[d, "|\cdot|"'] & \mathbb{G}_{\mathrm{m}} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \boldsymbol{\mu}_2 \arrow[r] \arrow[d] & \mathbb{G}_{\mathrm{m}} \arrow[r, "2"] \arrow[d] & \mathbb{G}_{\mathrm{m}} \arrow[r] & 1 \\
 & 1 & 1 & &
\end{tikzcd}$$

*of group schemes in the étale topology on $X$.*

*c) There's an exact sequence,*

$$1 \to \mathbb{G}_{\mathrm{m}} \to \mathbf{GO}(\mathscr{E}, b, \mathscr{L}) \xrightarrow{u} \mathbf{SO}(u(\mathscr{E}, b, \mathscr{L})) \to 1,$$

*and a commutative diagram with exact rows and columns,*

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
1 \arrow[r] & \boldsymbol{\mu}_2 \arrow[r] \arrow[d] & \mathbb{G}_{\mathrm{m}} \arrow[r] \arrow[d] & \mathbb{G}_{\mathrm{m}} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathbf{O} \arrow[r] \arrow[d, "u|_{\mathbf{O}}"'] & \mathbf{GO} \arrow[r, "\mu"] \arrow[d, "u"'] & \mathbb{G}_{\mathrm{m}} \arrow[r] & 1 \\
 & \mathbf{SO} \arrow[r, equal] \arrow[d] & \mathbf{SO} \arrow[d] & & \\
 & 1 & 1 & &
\end{tikzcd}$$

*of group schemes in the étale topology on $X$.*

*d) There's a canonical choice of group scheme isomorphism,*

$$\mathbf{SO}(\mathscr{E}, b, \mathscr{L}) \xrightarrow{\sim} \mathbf{SO}(u(\mathscr{E}, b, \mathscr{L})),$$

*with respect to which, the product homomorphism,*

$$\mathbf{GO}(\mathscr{E}, b, \mathscr{L}) \xrightarrow{|\cdot|\times u} \mathbb{G}_{\mathrm{m}} \times \mathbf{SO}(u(\mathscr{E}, b, \mathscr{L})),$$

*is an inverse to the scalar multiplication homomorphism,*

$$\mathbb{G}_{\mathrm{m}} \times \mathbf{SO}(\mathscr{E}, q, \mathscr{L}) \xrightarrow{m} \mathbf{GO}(\mathscr{E}, q, \mathscr{L}).$$

*In particular, these are isomorphisms.*

*Proof.* For the sequence in *b)*, $|\cdot|$ is an epimorphism since it has a section. By formulas (1.8), the kernel of $|\cdot|$ is contained in the kernels of both $\mu$ and $\det$, i.e. contained in $\mathbf{SO}(\mathscr{E}, b, \mathscr{L})$, and conversely, if $\mu$ and $\det$ are both trivial, then a square and an odd power of $|\cdot|$ is trivial, i.e. $|\cdot|$ is trivial.

The exactness of the sequence in *c)* is a consequence of the exactness of the sequence in *b)* and the isomorphism in *d)*. The commutativity of both diagrams follows from Proposition 1.17.

As for *d)*, we refer to Proposition 1.25. Once the canonical isomorphism is established, the rest follows. $\square$

### 1.3.2 Forms of rank one

An $\mathscr{L}$-valued bilinear space $(\mathscr{E}, b, \mathscr{L})$ of rank 1 is necessarily symmetric and defines a choice of square root $b : \mathscr{E} \otimes \mathscr{E} \xrightarrow{\sim} \mathscr{L}$ of the line bundle $\mathscr{L}$. When $\mathscr{L} = \mathscr{O}_X$, such spaces are called *square classes* in Knebusch [30] and *discriminant modules* in Knus [34]. By Milne [36, III §4], the set of isometry classes of discriminant modules over $X$ is in bijection with $H^1_{\text{ét}}(X, \boldsymbol{\mu}_2)$, which also follows from Theorem 1.4 in light of the comparison between étale cohomology and Čech cohomology, see Appendix A.

General $\mathscr{L}$-valued spaces of rank 1 are called *twisted discriminant bundles* in Balaji [2]. We will call them *$\mathscr{L}$-valued lines* or simply *$\mathscr{L}$-lines*. For a fixed $\mathscr{L}$, the set of isometry classes of $\mathscr{L}$-valued lines is an $H^1_{\text{ét}}(X, \boldsymbol{\mu}_2)$-principal homogeneous space. Note that the similarity class of an $\mathscr{L}$-valued line $(\mathscr{E}, b, \mathscr{L})$ is uniquely determined by the isomorphism class of the line bundle $\mathscr{E}$.

The special orthogonal group $\mathbf{SO}(\mathscr{E}, q, \mathscr{L})$ of an $\mathscr{L}$-valued line $(\mathscr{E}, b, \mathscr{L})$ is trivial and under the identifications $\mathbf{O}(\mathscr{E}, q, \mathscr{L}) = \boldsymbol{\mu}_2$ and $\mathbf{GO}(\mathscr{E}, q, \mathscr{L}) = \mathbb{G}_{\mathrm{m}}$, the multiplier sequence (1.5) coincides with the Kummer sequence,

$$1 \to \boldsymbol{\mu}_2 \to \mathbb{G}_{\mathrm{m}} \xrightarrow{2} \mathbb{G}_{\mathrm{m}} \to 1.$$

#### Dual line

For an $\mathscr{L}$-valued line $(\mathscr{N}, l, \mathscr{L})$ define the *dual line* $(\mathscr{N}, l, \mathscr{L})^\vee = (\mathscr{N}^\vee, l^\vee, \mathscr{L}^\vee)$ by the composition (and abuse of notation),

$$l^\vee : \mathscr{N}^\vee \otimes \mathscr{N}^\vee \xrightarrow{\text{can}} (\mathscr{N} \otimes \mathscr{N})^\vee \xrightarrow{l^{-1\vee}} \mathscr{L}^\vee,$$

using the canonical isomorphism given by

$$\begin{array}{rcl} \text{can} : \mathscr{N}^\vee \otimes \mathscr{N}^\vee & \to & (\mathscr{N} \otimes \mathscr{N})^\vee \\ f \otimes g & \mapsto & n \otimes m \mapsto f(n)\, g(m) \end{array}$$

on section over $U \to X$, multiplying sections via the standard $\mathscr{O}_X$-line $\mathscr{O}_X \otimes \mathscr{O}_X \to \mathscr{O}_X$.

#### Scaling

**Definition 1.24.** Let $(\mathscr{E}, b, \mathscr{L})$ be an $\mathscr{L}$-valued bilinear form and $(\mathscr{N}, l, \mathscr{L})$ an $\mathscr{L}$-line. Then the form $(\mathscr{E}, b, \mathscr{L})$ *scaled by* $(\mathscr{N}, l, \mathscr{L})$, or $(\mathscr{E}, b, \mathscr{L})/(\mathscr{N}, l, \mathscr{L})$ is by definition the $\mathscr{O}_X$-valued bilinear form $(\mathscr{N}^\vee \otimes \mathscr{E}, b/l, \mathscr{O}_X)$ given by

$$b/l : (\mathscr{N}^\vee \otimes \mathscr{E}) \otimes (\mathscr{N}^\vee \otimes \mathscr{E}) \xrightarrow{l^{-1\vee} \otimes b} \mathscr{L}^\vee \otimes \mathscr{L} \xrightarrow{\text{can}} \mathscr{O}_X.$$

Note that we've already utilized the scaling construction to define the discriminant and normalized forms. If a bilinear space $(\mathscr{E}, b, \mathscr{L})$ can be scaled by an $\mathscr{L}$-line, then necessarily the line bundle $\mathscr{L}$ is a square in the Picard group of $X$. The similarity classes of possible scalings of $(\mathscr{E}, b, \mathscr{L})$ are in bijection with the isomorphism classes of line bundles of order $\leq 2$. For bilinear forms of odd rank, the associated normalized form (see §1.3.1) is a canonical choice of scaling.

**Isometry groups of scaled forms**

The following is in analogy with well-known facts about general linear groups of vector bundles, see Remark 1.3.

**Proposition 1.25.** *Let $(\mathscr{E}, b, \mathscr{L})$ be a symmetric bilinear space of rank $n$ on $X$.*

*a) The group schemes $\mathbf{O}(\mathscr{E}, b, \mathscr{L})$ and $\mathbf{O}(\mathscr{E}', b', \mathscr{L}')$ and the group schemes $\mathbf{GO}(\mathscr{E}, b, \mathscr{L})$ and $\mathbf{GO}(\mathscr{E}', b', \mathscr{L}')$ are isomorphic if and only if there exists a line bundle $\mathscr{N}$ such that $\mathscr{E}' \cong \mathscr{N} \otimes \mathscr{E}$ and $\mathscr{L}' \cong \mathscr{N}^{\otimes 2} \otimes \mathscr{L}$.*

*b) In particular, if $\mathscr{L}$ is a square in $\operatorname{Pic}(X)$, then the choice of $\mathscr{L}$-valued line $(\mathscr{N}, l, \mathscr{L})$ yields an induced group scheme isomorphism $\mathbf{O}(\mathscr{E}, b, \mathscr{L}) \xrightarrow{\sim} \mathbf{O}((\mathscr{E}, b, \mathscr{L})/(\mathscr{N}, l, \mathscr{L}))$.*

*c) The class of $\mathbf{O}(\mathscr{E}, b, \mathscr{L})$ in $\check{H}^1_{\text{ét}}(X, \mathbf{Aut}(\mathbf{O}_n))$ arises from the image of the inner automorphism map $\mathbf{O}_n \to \mathbf{Aut}(\mathbf{O}_n)$ if and only if $\mathscr{L}$ is a square in $\operatorname{Pic}(X)$.*

*Proof.* We will prove *b)*, and then *a)* is similar. For *b)*, choosing an isomorphism $\varphi : \mathscr{N} \otimes \mathscr{N} \xrightarrow{\sim} \mathscr{L}^\vee$ of line bundles, define the following natural vector bundle isomorphism

$$\begin{array}{rcl} \varphi_* \,:\, \mathscr{N} \otimes \mathscr{H}om(\mathscr{E}, \mathscr{L}) & \xrightarrow{\sim} & (\mathscr{N} \otimes \mathscr{E})^\vee \\ n \otimes g & \mapsto & n' \otimes w \mapsto \varphi(n \otimes n')(g(w)). \end{array}$$

Now we'll show that we have the equality

$$\psi_{\varphi \otimes b} = \varphi_* \circ \mathrm{id}_{\mathscr{N}} \otimes \psi_b \;:\; \mathscr{N} \otimes \mathscr{E} \xrightarrow{\sim} \mathscr{H}om(\mathscr{N} \otimes \mathscr{E}, \mathscr{O}_X).$$

Indeed, we compute

$$\begin{array}{rcl} \psi_{\varphi \otimes b} \,:\, \mathscr{N} \otimes \mathscr{E} & \xrightarrow{\sim} & (\mathscr{N} \otimes \mathscr{E})^\vee \\ n \otimes v & \mapsto & n' \otimes w \mapsto (\varphi \otimes b)(n \otimes v, n' \otimes w) = \varphi(n \otimes n')(b(v, w)), \end{array}$$

and

$$\begin{array}{rcccl} \varphi_* \circ \mathrm{id}_{\mathscr{N}} \otimes \psi_b \,:\, \mathscr{N} \otimes \mathscr{E} & \xrightarrow{\sim} & \mathscr{N} \otimes \mathscr{H}om(\mathscr{E}, \mathscr{L}) & \xrightarrow{\sim} & (\mathscr{N} \otimes \mathscr{E})^\vee \\ n \otimes v & \mapsto & w \mapsto n \otimes b(v, w) & \mapsto & n' \otimes w \mapsto \varphi(n \otimes n')(b(v, w)). \end{array}$$

Now define a morphism of sheaves $\Phi : \mathbf{O}(\mathscr{E}, b, \mathscr{L}) \xrightarrow{\sim} \mathbf{O}(\mathscr{N} \otimes \mathscr{E}, \varphi \otimes b, \mathscr{O}_X)$ by

$$\begin{array}{rcl} \Phi(U) \,:\, \mathbf{O}(\mathscr{E}, b, \mathscr{L})(U) & \to & \mathbf{O}(\mathscr{N} \otimes \mathscr{E}, \varphi \otimes b, \mathscr{O}_X)(U) \\ f : \mathscr{E}|_U \to \mathscr{E}|_U & \mapsto & \mathrm{id}_{\mathscr{N}|_U} \otimes f : (\mathscr{N} \otimes \mathscr{E})|_U \to (\mathscr{N} \otimes \mathscr{E})|_U. \end{array}$$

To see that $\Phi$ is well defined, we need to check that

$$(\mathrm{id}_{\mathscr{N}|_U} \otimes f)^\vee \circ \psi_{\varphi \otimes b}|_U \circ (\mathrm{id}_{\mathscr{N}|_U} \otimes f) = \psi_{\varphi \otimes b}|_U,$$

and to this end we compute, for $n \otimes v \in (\mathscr{N} \otimes \mathscr{E})(U)$,

$$\begin{array}{rl} \multicolumn{2}{l}{(\mathrm{id}_{\mathscr{N}|_U U} \otimes f)^\vee \circ \psi_{\varphi \otimes b}|_U \circ (\mathrm{id}_{\mathscr{N}|_U} \otimes f)(n \otimes v)} \\ = & (\mathrm{id}_{\mathscr{N}|_U} \otimes f)^\vee (n \otimes w \mapsto \varphi(n \otimes n') \otimes b|_U(f(v), w)) \\ = & n' \otimes w \mapsto \varphi(n \otimes n') \otimes b|_U(f(v), f(w)) \\ = & n' \otimes w \mapsto \varphi(n \otimes n') \otimes b|_U(v, w) \\ = & \psi_{\varphi \otimes b}|_U(n, v), \end{array}$$

using the fact that $f \in \mathbf{O}(\mathscr{E}, b, \mathscr{L})(U)$. Thus $\Phi$ is in fact a morphism of sheaves, and is clearly a bijective homomorphism on sections, thus $\Phi$ is an isomorphism of sheaves of groups. Now *c)* is a consequence of *b)* and results from Appendix A. $\square$

## 1.4 Metabolic forms

The notion of a metabolic form – the correct generalization to schemes of the notion of hyperbolic form – is necessary to the construction of the Grothendieck-Witt and Witt groups of schemes. The formalism of $\mathscr{O}_X$-valued metabolic forms over arbitrary schemes was first introduced by Knebusch [30, I §3] and then extended to the context of exact categories with duality by Balmer [3], [4], and [5]. We assume that $X$ is a scheme with $\frac{1}{2} \in \mathscr{O}_X$.

### 1.4.1 Orthogonal complements

We will summarize the formalism of metabolic forms in the context of $\mathscr{L}$-valued forms, i.e. the exact category of vector bundles on a scheme with duality given by the functor $(-)^{\vee_{\mathscr{L}}} = \mathscr{H}om(-, \mathscr{L})$.

**Definition 1.26.** Let $(\mathscr{E}, b, \mathscr{L})$ be a bilinear form on $X$ and $\mathscr{V} \subset \mathscr{E}$ be a vector subbundle, i.e. suppose there's an exact sequence,

$$0 \to \mathscr{V} \xrightarrow{j} \mathscr{E} \xrightarrow{p} \mathscr{E}/\mathscr{V} \to 0,$$

of vector bundles on $X$. Then define the *($\mathscr{L}$-valued) orthogonal complement* $\mathscr{V}^\perp \subset \mathscr{E}$ as the kernel of the composition,

$$\mathscr{E} \xrightarrow{\psi_b} \mathscr{H}om(\mathscr{E}, \mathscr{L}) \xrightarrow{j^{\vee_{\mathscr{L}}}} \mathscr{H}om(\mathscr{V}, \mathscr{L}) \to 0.$$

i.e. via the exact sequence,

$$0 \to \mathscr{V}^\perp \xrightarrow{j^\perp} \mathscr{E} \xrightarrow{j^{\vee_{\mathscr{L}}} \circ \psi_b} \mathscr{H}om(\mathscr{V}, \mathscr{L}) \to 0,$$

of vector bundles on $X$.

By definition, there's a canonical isomorphism,

$$\mathscr{V}^\perp \xrightarrow{\psi_b|_{\mathscr{V}^\perp}} \mathscr{H}om(\mathscr{E}/\mathscr{V}, \mathscr{L}),$$

fitting into the following commutative diagram with exact rows,

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{V}^\perp \arrow[r, "j^\perp"] \arrow[d, "\psi_b|_{\mathscr{V}^\perp}"'] & \mathscr{E} \arrow[r, "j^{\vee_{\mathscr{L}}} \circ \psi_b"] \arrow[d, "\psi_b"] & \mathscr{H}om(\mathscr{V}, \mathscr{L}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathscr{H}om(\mathscr{E}/\mathscr{V}, \mathscr{L}) \arrow[r, "p^{\vee_{\mathscr{L}}}"] & \mathscr{H}om(\mathscr{E}, \mathscr{L}) \arrow[r, "j^{\vee_{\mathscr{L}}}"] & \mathscr{H}om(\mathscr{V}, \mathscr{L}) \arrow[r] & 0.
\end{tikzcd}$$

Furthermore, there's a canonical commutative square,

$$\begin{tikzcd} \mathscr{V}^{\perp\perp} \arrow[r, equal] \arrow[d, "\psi_b|_{\mathscr{V}^{\perp\perp}}"'] & \mathscr{V} \arrow[d, "\mathrm{can}^{\mathscr{L}}"] \\ \mathscr{H}om(\mathscr{E}/\mathscr{V}^{\perp}, \mathscr{L}) \arrow[r, "(j^{\vee\mathscr{L}} \circ \psi_b)^{\vee\mathscr{L}}"] & \mathscr{H}om(\mathscr{H}om(\mathscr{V}, \mathscr{L}), \mathscr{L}) \end{tikzcd}$$

and the following commutative diagrams,

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{V}^{\perp} \arrow[r, "j^{\perp}"] \arrow[d, "\psi_b|_{\mathscr{V}^{\perp}}"'] & \mathscr{E} \arrow[d, "\psi_b"] \\ 0 \arrow[r] & \mathscr{H}om(\mathscr{E}/\mathscr{V}, \mathscr{L}) \arrow[r, "p^{\vee\mathscr{L}}"] & \mathscr{H}om(\mathscr{E}, \mathscr{L}) \end{tikzcd} \qquad \begin{tikzcd} \mathscr{E} \arrow[r] \arrow[d, "\psi_b"'] & \mathscr{E}/\mathscr{V}^{\perp} \arrow[r] \arrow[d, "j^{\vee\mathscr{L}} \circ \psi_b"] & 0 \\ \mathscr{H}om(\mathscr{E}, \mathscr{L}) \arrow[r, "j^{\vee\mathscr{L}}"] & \mathscr{H}om(\mathscr{V}, \mathscr{L}) \arrow[r] & 0 \end{tikzcd}$$

of vector bundles on $X$.

### 1.4.2 Metabolic forms

**Definition 1.27.** Let $(\mathscr{E}, b, \mathscr{L})$ be a bilinear form on $X$ and $\mathscr{V} \subset \mathscr{E}$ a vector subbundle. Then $\mathscr{V}$ is called *isotropic* or an *($\mathscr{L}$-valued) sublagrangian* if $\mathscr{V} \subset \mathscr{V}^{\perp}$ and an *($\mathscr{L}$-valued) lagrangian* if $\mathscr{V} = \mathscr{V}^{\perp}$, i.e. if there's an isomorphism of short exact sequences,

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{V} \arrow[r, "j = j^{\perp}"] \arrow[d, "\mathrm{can}^{\mathscr{L}}"'] & \mathscr{E} \arrow[r, "j^{\vee\mathscr{L}} \circ \psi_b"] \arrow[d, "\psi_b"] & \mathscr{H}om(\mathscr{V}, \mathscr{L}) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathscr{H}om(\mathscr{H}om(\mathscr{V}, \mathscr{L}), \mathscr{L}) \arrow[r, "(j^{\vee\mathscr{L}} \circ \psi_b)^{\vee\mathscr{L}}"] & \mathscr{H}om(\mathscr{E}, \mathscr{L}) \arrow[r, "j^{\vee\mathscr{L}}"] & \mathscr{H}om(\mathscr{V}, \mathscr{L}) \arrow[r] & 0 \end{tikzcd}$$

of vector bundles on $X$. A bilinear form on $X$ is called *metabolic* if it has a lagrangian. A bilinear form on $X$ containing no sublagrangians is called *anisotropic*.

### 1.4.3 Split metabolic forms

A matrix of $\mathscr{O}_X$-module morphisms,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \begin{array}{ll} a : \mathscr{V} \to \mathscr{H}om(\mathscr{V}, \mathscr{L}) & b : \mathscr{V}' \to \mathscr{H}om(\mathscr{V}, \mathscr{L}) \\ c : \mathscr{V} \to \mathscr{H}om(\mathscr{V}', \mathscr{L}) & d : \mathscr{V}' \to \mathscr{H}om(\mathscr{V}', \mathscr{L}) \end{array}$$

defines a bilinear form on $\mathscr{V} \oplus \mathscr{V}'$ in the usual way.

A metabolic bilinear form $(\mathscr{E}, b, \mathscr{L})$ is called a *split metabolic* form if the exact sequence defining a lagrangian,

$$0 \to \mathscr{V} \to \mathscr{E} \to \mathscr{H}om(\mathscr{V}, \mathscr{L}) \to 0,$$

is a split exact sequence of vector bundles. In this case, $(\mathscr{E}, b, \mathscr{L})$ is isometric to

$$M(\mathscr{V}, \psi, \mathscr{L}) = (\mathscr{V} \oplus \mathscr{H}om(\mathscr{V}, \mathscr{L}), \begin{pmatrix} \psi & \mathrm{id} \\ \mathrm{can}^{\mathscr{L}} & 0 \end{pmatrix}, \mathscr{L}),$$

for some $\mathscr{O}_X$-module morphism $\psi : \mathscr{V} \to \mathscr{H}om(\mathscr{V}, \mathscr{L})$. Then $M(\mathscr{V}, \psi, \mathscr{L})$ is always non-singular and is symmetric if and only if $\psi = \psi^t$ is symmetric (see formula 1.3). The split metabolic form with $\psi = 0$ is called a *hyperbolic* form,

$$H_{\mathscr{L}}(\mathscr{V}) = (\mathscr{V} \oplus \mathscr{H}om(\mathscr{V}, \mathscr{L}), \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{can}^{\mathscr{L}} & 0 \end{pmatrix}, \mathscr{L}).$$

If $M(\mathscr{V}, \psi, \mathscr{L})$ is a symmetric split metabolic form then the map

$$\begin{pmatrix} \mathrm{id} & 0 \\ -\frac{1}{2}\psi & \mathrm{id} \end{pmatrix} : H_{\mathscr{L}}(\mathscr{V}) \to M(\mathscr{V}, a, \mathscr{L})$$

is an isometry to a hyperbolic form, under our assumption that $\frac{1}{2} \in \mathscr{O}_X$. Over an affine scheme, every metabolic form is split metabolic, which follows by adapting Knebusch [30, I §3, Corollary 1].

### 1.4.4 Basic properties of metabolic forms

Metabolic forms have good properties with respect to tensor products. These properties can be adapted from Knebusch [30, I §3].

**Proposition 1.28.** *Let $(\mathscr{E}, b, \mathscr{L})$ and $(\mathscr{E}', b', \mathscr{L}')$ be bilinear spaces on $X$.*

*a) If $\mathscr{V} \subset \mathscr{E}$ be a (sub)lagrangian, then $\mathscr{V} \otimes \mathscr{E}' \subset \mathscr{E} \otimes \mathscr{E}'$ is a (sub)lagrangian of $(\mathscr{E}, b, \mathscr{L}) \otimes (\mathscr{E}', b', \mathscr{L}')$.*

*b) In particular, we have isometries*

$$M(\mathscr{V}, \psi, \mathscr{L}) \otimes (\mathscr{E}', b', \mathscr{L}') \cong M(\mathscr{V} \otimes \mathscr{E}', \psi \otimes \psi_{b'}, \mathscr{L} \otimes \mathscr{L}')$$
$$H_{\mathscr{L}}(\mathscr{V}) \otimes (\mathscr{E}', b', \mathscr{L}') \cong H_{\mathscr{L} \otimes \mathscr{L}'}(\mathscr{V} \otimes \mathscr{E}').$$

*c) We have an isometry $(\mathscr{E}, b, \mathscr{L}) \perp (\mathscr{E}, -b, \mathscr{L}) \cong M(\mathscr{E}, \psi_b)$*

### 1.4.5 Splitting principle for metabolic forms

In analogy with the classical splitting principle for vector bundles, there's a splitting principle for metabolic bundles, see Fulton [19, §2] for a treatment using isotropic flag bundles, or Esnault/Kahn/Viehweg [17, §5] using projective bundles.

**Theorem 1.29.** *Let $(\mathscr{E}, b, \mathscr{L})$ be a metabolic form on $X$ with lagrangian $\mathscr{V}$. Let $H(X)$ mean either coherent, $\ell$-adic, étale, or Chow cohomology.*

*a) There exists a morphism $f : Y \to X$ such that $f^* : H(X) \to H(X)$ an isomorphism and so that $f^*(\mathscr{E}, b, \mathscr{L})$ is a split metabolic form with lagrangian $f^*\mathscr{V}$.*

*b) There exists a morphism $f : Y \to X$ such that $f^* : H(X) \to H(Y)$ injective and so that $f^*(\mathscr{E}, b, \mathscr{L})$ is isometric to an orthogonal sum of hyperbolic planes*

$$H_{f^*\mathscr{L}}(\mathscr{V}_1) \perp \cdots \perp H_{f^*\mathscr{L}}(\mathscr{V}_m),$$

*for line bundles $\mathscr{V}_1, \ldots, \mathscr{V}_m$ on $Y$.*

*Proof.* As for *a)*, this is a classical construction. Following Fulton [19, §2], let $\mathscr{V} \xrightarrow{j} \mathscr{E}$ be a lagrangian and let $P \subset \mathscr{H}om(\mathscr{E}, \mathscr{V})$ be the subbundle whose sections over $U \to X$ are $\mathscr{O}_U$-module morphisms $\varphi : \mathscr{E}|_U \to \mathscr{V}|_U$ such that $j|_U \circ \varphi = \mathrm{id}_U$. Let $Y = \mathbb{V}(P)$ be the corresponding affine bundle, and $f : Y \to X$ be the natural projection. Then $f^*\mathscr{E}$ has a tautological projection to $f^*\mathscr{V}$, hence $f^*(\mathscr{E}, b, \mathscr{L})$ is a split metabolic space with lagrangian $f^*\mathscr{V}$. The fact that $f^*$ is an isomorphism on cohomology for an affine bundle is standard.

As for *b)*, see Fulton [19, §2] for a construction of the isotropic flag bundle of a metabolic bundle. Pulling back to this flag variety yields the required splitting. □

**Witt cancellation**

The analogue of Witt cancellation on schemes only holds only "up to metabolic forms," see Knebusch [30, I §4, Theorem 3].

**Proposition 1.30.** *Let $(\mathscr{E}, b, \mathscr{L})$ be a symmetric bilinear space and $\mathscr{V} \subset \mathscr{E}$ a sublagrangian. Then the subbundle $\mathscr{V}^\perp$, via the diagonal inclusion,*

$$\mathscr{V}^\perp \to \mathscr{E} \oplus \mathscr{V}^\perp/\mathscr{V},$$

*is a lagrangian for the symmetric bilinear space $(\mathscr{E}, b, \mathscr{L}) \perp (\mathscr{V}^\perp/\mathscr{V}, -\bar{b}, \mathscr{L})$.*

The Witt group is then constructed as the quotient of the free abelian group (under orthogonal sum) of symmetric bilinear forms by the subgroup of metabolic spaces, not just split metabolic spaces (as in the affine case). Thus according to Proposition 1.30, every symmetric bilinear form is equivalent in the Witt group to an anisotropic form.

## 1.5 The Clifford algebra

The Clifford algebra of an $\mathscr{O}_X$-valued symmetric bilinear form over a scheme $X$ with $\frac{1}{2} \in \mathscr{O}_X$ is a locally free sheaf of $\mathscr{O}_X$-algebras, which is gotten by sheafifying the construction for forms over rings. All the standard properties of the Clifford algebra can then be generalized to $\mathscr{O}_X$-valued forms over schemes. For symmetric bilinear forms with valued in a non-trivial line bundle $\mathscr{L}$ (especially if $\mathscr{L}$ is not a square in the Picard group), another construction is required. We follow the construction (over affine schemes) of Bichsel/Knus [7] and Caenepeel/Van Oysaeyen [8]. The *generalized Clifford algebra* of an $\mathscr{L}$-valued symmetric bilinear form is defined by base change to the punctured total space $\mathbb{V}^\bullet(\mathscr{L})$ of the the line bundle $\mathscr{L}$, and then applying the standard Clifford algebra functor to the resulting ($\mathscr{O}_{\mathbb{V}^\bullet(\mathscr{L})}$-valued) form. We also utilize the construction of an even Clifford group of an $\mathscr{L}$-valued symmetric bilinear space.

### 1.5.1 The classical Clifford algebra

For the moment let $X$ be any scheme (even with 2 not invertible). Let $(\mathscr{E}, q, \mathscr{O}_X)$ be a fixed $\mathscr{O}_X$-valued quadratic form over $X$. The *Clifford algebra* $\mathscr{C}(\mathscr{E}, q, \mathscr{O}_X)$ is the sheaf of $\mathscr{O}_X$-algebras associated to the presheaf,

$$U \mapsto T\mathscr{E}(U)/\langle v \otimes v - q(v) : v \in \mathscr{E}(U) \rangle$$

of the tensor algebra modulo the two-sided ideal generated by $v \otimes v - q(v)$ for $v \in \mathscr{E}(U)$. There's a canonical $\mathscr{O}_X$-module morphism $i : \mathscr{E} \to \mathscr{C}(\mathscr{E}, q, \mathscr{O}_X)$. The Clifford algebra is functorial for pull backs and satisfies the following universal property. For any sheaf of $\mathscr{O}_X$-algebras $\mathscr{C}'$ and $\mathscr{O}_X$-module morphism $f : \mathscr{E} \to \mathscr{C}'$ such that the following diagram of maps of sheaves commutes,

$$\begin{tikzcd} \mathscr{E} \arrow[r, "f"] \arrow[d, "q"'] & \mathscr{C}' \arrow[d, "(-)^2"] \\ \mathscr{O}_X \arrow[r, "1_{\mathscr{C}'}"] & \mathscr{C}' \end{tikzcd}$$

there exists a unique $\mathscr{O}_X$-algebra homomorphism $c : \mathscr{C}(\mathscr{E}, q, \mathscr{O}_X) \to \mathscr{C}'$ such that $c \circ i = f$.

**Lemma 1.31.** *Let $(\mathscr{E}, q, \mathscr{O}_X)$ be a quadratic form of rank $n$ over $X$. Then its Clifford algebra is a locally free $\mathscr{O}_X$-algebra of rank $2^n$ and the canonical map $i : \mathscr{E} \to \mathscr{C}(\mathscr{E}, q, \mathscr{O}_X)$ is a monomorphism.*

*Proof.* Localizing we may assume $X$ is the spectrum of a local ring and $\mathscr{E}$ is free, then we apply the classical Poincaré-Birkhoff-Witt theorem. $\square$

The Clifford algebra of an $\mathscr{O}_X$-valued symmetric bilinear form $(\mathscr{E}, b, \mathscr{O}_X)$ is defined to be the Clifford algebra of it's associated quadratic form, see §1.1.1 formula (1.2). Then we have the classical relations in the Clifford algebra

$$v \cdot v = b(v, v), \qquad v \cdot w + w \cdot v = 2b(v, w), \qquad \text{for } v, w \in \mathscr{E}(U)$$

on sections over $U \to X$, where the first formula is the defining relation of the Clifford algebra and the second is implied from it (and equivalent if $\frac{1}{2} \in \mathscr{O}_X$).

### Even Clifford algebra and center

We now assume that $\frac{1}{2} \in \mathscr{O}_X$ and that $(\mathscr{E}, b, \mathscr{O}_X)$ is an $\mathscr{O}_X$-valued bilinear space. The Clifford algebra $\mathscr{C} = \mathscr{C}(\mathscr{E}, b, \mathscr{O}_X)$ has a $\mathbb{Z}/2\mathbb{Z}$-grading $\mathscr{C} = \mathscr{C}_0 \oplus \mathscr{C}_1$ into *even* and *odd* parts, inherited from the even and odd parts of the tensor algebra. The following properties are straightforward generalizations from the case of affine schemes, see Knus [34, IV §2-3].

**Proposition 1.32.** *Let $\mathscr{C}$ be the Clifford algebra of a quadratic space of rank $n$ on $X$.*

*a) If $n$ is even, then $\mathscr{C}$ is an Azumaya algebra over $X$. If $\mathscr{Z}$ is the center of the even Clifford algebra $\mathscr{C}_0$, then $\mathscr{Z}$ is a rank 2 étale $\mathscr{O}_X$-algebra and $\mathscr{C}_0$ has the structure of an Azumaya algebra over $\mathbf{Spec}\,\mathscr{Z}$.*

*b) If $n$ is odd, then $\mathscr{C}_0$ is an Azumaya algebra over $X$. If $\mathscr{Z}$ is the center of the Clifford algebra $\mathscr{C}$, then $\mathscr{Z}$ is a rank 2 étale $\mathscr{O}_X$-algebra and $\mathscr{C}$ has the structure of an Azumaya algebra over $\mathbf{Spec}\,\mathscr{Z}$. Finally, $\mathscr{Z} = \mathscr{Z}_0 \oplus \mathscr{Z}_1$ inherits a $\mathbb{Z}/2\mathbb{Z}$-grading, with $\mathscr{Z} = \mathscr{O}_X \cdot 1_{\mathscr{C}}$ and the multiplication in the Clifford algebra induces an $\mathscr{O}_X$-valued line $\mathscr{Z}_1 \otimes \mathscr{Z}_1 \xrightarrow{\sim} \mathscr{O}_X$ on the line bundle $\mathscr{Z}_1$ over $X$.*

**Definition 1.33.** Let $(\mathscr{E}, b, \mathscr{O}_X)$ be an $\mathscr{O}_X$-valued symmetric bilinear space of rank $n$ on $X$. Define the *Arf invariant* $\mathrm{Arf}(\mathscr{E}, b, \mathscr{O}_X)$ to be the isomorphism class of the rank 2 étale $\mathscr{O}_X$-algebra

$$\mathscr{Z} = \mathscr{Z}(\mathscr{E}, b, \mathscr{O}_X) = \begin{cases} \text{center of } \mathscr{C}_0(\mathscr{E}, b, \mathscr{O}_X) & \text{if } n \text{ is even} \\ \text{center of } \mathscr{C}(\mathscr{E}, b, \mathscr{O}_X) & \text{if } n \text{ is odd} \end{cases}$$

Define the *Clifford invariant* $c(\mathscr{E}, b, \mathscr{O}_X)$ the be the isomorphism class of the Azumaya algebra

$$\begin{cases} \mathscr{C}(\mathscr{E}, b, \mathscr{O}_X) & \text{if } n \text{ is even} \\ \mathscr{C}_0(\mathscr{E}, b, \mathscr{O}_X) & \text{if } n \text{ is odd} \end{cases}$$

in the Brauer group of $X$.

**Lemma 1.34.** *Let $(\mathscr{E}, b, \mathscr{L})$ be a symmetric bilinear form of rank $n$ over $X$. Then we have the equation,*

$$\mathrm{Arf}(\mathscr{E}, b, \mathscr{O}_X) = (-1)^{\frac{n(n-1)}{2}} + \det(\mathscr{E}, b, \mathscr{O}_X),$$

*interpreted as classes in $H^1_{\text{ét}}(X, \boldsymbol{\mu}_2)$.*

### Functoriality of the Clifford algebra

By the universal property, the Clifford algebra is functorial with respect to isometries. If $\varphi : (\mathscr{E}, b, \mathscr{O}_X) \to (\mathscr{E}', b', \mathscr{O}_X)$ is an isometry, then there's an induced ($\mathbb{Z}/2\mathbb{Z}$-)graded $\mathscr{O}_X$-algebra isomorphism $\mathscr{C}(\varphi) : \mathscr{C}(\mathscr{E}, b, \mathscr{O}_X) \to \mathscr{C}(\mathscr{E}', b', \mathscr{O}_X)$ making the following diagram

$$\begin{tikzcd} \mathscr{E} \arrow[r, "\varphi"] \arrow[d, "i"'] & \mathscr{E}' \arrow[d, "i'"] \\ \mathscr{C}(\mathscr{E}, b, \mathscr{O}_X) \arrow[r, "\mathscr{C}(\varphi)"] & \mathscr{C}(\mathscr{E}', b', \mathscr{O}_X) \end{tikzcd}$$

commutative. The Clifford algebra isomorphism $\mathscr{C}(\varphi)$ restricts to an $\mathscr{O}_X$-algebra isomorphism $\mathscr{C}(\varphi)|_{\mathscr{Z}} : \mathscr{Z}(\mathscr{E}, b, \mathscr{O}_X) \to \mathscr{Z}(\mathscr{E}', b', \mathscr{O}_X)$ of the corresponding centers. Functoriality with respect to similarity transformations is more subtle, see Knus [34, IV §7].

**Proposition 1.35.** *Let $(\varphi, \mu_\varphi) : (\mathscr{E}, b, \mathscr{O}_X) \to (\mathscr{E}', b', \mathscr{O}_X)$ be a similarity transformation of quadratic spaces on $X$. Then there's a unique $\mathscr{O}_X$-algebra isomorphism*

$$\mathscr{C}_0(\varphi, \mu_\varphi) : \mathscr{C}_0(\mathscr{E}, b, \mathscr{O}_X) \to \mathscr{C}_0(\mathscr{E}', b', \mathscr{O}_X)$$

*making the following diagram*

$$\begin{tikzcd} \mathscr{E} \otimes \mathscr{E} \arrow[r, "\varphi \otimes \varphi"] \arrow[d, "i \otimes i"'] & \mathscr{E}' \otimes \mathscr{E}' \arrow[d, "i' \otimes i'"] \\ \mathscr{C}_0(\mathscr{E}, b, \mathscr{O}_X) \arrow[r, "\mu_\varphi \cdot \mathscr{C}_0(\varphi, \mu_\varphi)"] & \mathscr{C}_0(\mathscr{E}', b', \mathscr{O}_X) \end{tikzcd}$$

*commutative, where $\mu_\varphi : \mathscr{O}_X \to \mathscr{O}_X$ induces a unique $\mathscr{O}_X$-algebra isomorphism in the usual way. There's also a unique $\mathscr{O}_X$-module isomorphism*

$$\mathscr{C}_1(\varphi, \mu_\varphi) : \mathscr{C}_1(\mathscr{E}, b, \mathscr{O}_X) \to \mathscr{C}_1(\mathscr{E}', b', \mathscr{O}_X),$$

*which is $\mathscr{C}_0(\varphi, \mu_\varphi)$-semilinear.*

**Clifford group**

The isometry $-\mathrm{id}$ on any quadratic space $(\mathscr{E}, b, \mathscr{O}_X)$ induces the *standard automorphism* $I = \mathscr{C}(-\mathrm{id})$ on the Clifford algebra, which acts as $(-1)^{\varepsilon}\mathrm{id}$ on $\mathscr{C}_{\varepsilon}$, for $\varepsilon \in \{0, 1\}$. There is also a *canonical involution* $\sigma$ on the Clifford algebra induced from the canonical antiautomorphism on the tensor algebra $v_1 \otimes \cdots \otimes v_n \mapsto v_n \otimes \cdots \otimes v_1$.

For any ($\mathbb{Z}/2\mathbb{Z}$-)graded sheaf of $\mathscr{O}_X$-algebras $\mathscr{A}$, denote by $\mathscr{A}_{lh}$ the presheaf of sets of locally homogeneous elements of $\mathscr{A}$, i.e. elements that are homogeneous when restricted to the local ring of any point.

The *Clifford group* $\mathbf{\Gamma}(\mathscr{E}, b, \mathscr{O}_X)$ is the sheaf of groups associated to the presheaf

$$U \mapsto \{x \in \mathscr{C}(\mathscr{E}, b, \mathscr{O}_X)_{lh}(U)^{\times} : x\mathscr{E}(U)I(x)^{-1} \subset \mathscr{E}(U)\},$$

and is represented as a group scheme in the étale topology on $X$. The Clifford group $\mathbf{\Gamma} = \mathbf{\Gamma}(\mathscr{E}, b, \mathscr{O}_X)$ inherits a disjoint union decomposition $\mathbf{\Gamma} = \mathbf{\Gamma}_0 \cup \mathbf{\Gamma}_1$ from the grading on the Clifford algebra (and the local homogeneity), and we call $\mathbf{\Gamma}_0 = \mathbf{S\Gamma}(\mathscr{E}, b, \mathscr{O}_X)$ the *even Clifford group*.

On sections over $U \to X$, the map defined by $N(x) = x\sigma(x)$ for $x \in \mathbf{\Gamma}(U)$, induces a homomorphism of group schemes

$$N : \mathbf{\Gamma}(\mathscr{E}, b, \mathscr{O}_X) \to \mathbb{G}_{\mathrm{m}}$$

called the *Clifford norm* and we define the *pin group* $\mathbf{Pin}(\mathscr{E}, b, \mathscr{O}_X)$ as its kernel, see Knus [34, IV §6]. The pin group is a smooth affine algebraic group scheme over $X$. There's an exact sequence of group schemes,

$$1 \to \mathbf{Pin}(\mathscr{E}, b, \mathscr{O}_X) \to \mathbf{\Gamma}(\mathscr{E}, b, \mathscr{O}_X) \xrightarrow{N} \mathbb{G}_{\mathrm{m}} \to 1, \tag{1.10}$$

in the étale topology on $X$.

On sections over $U \to X$, the map defined by $r(x) = v \mapsto xvI(x)^{-1}$ for $x \in \mathbf{\Gamma}(U)$ and $v \in \mathscr{E}(U) \hookrightarrow \mathscr{C}(U)$ induces a homomorphism of group schemes

$$r : \mathbf{\Gamma}(\mathscr{E}, b, \mathscr{O}_X) \to \mathbf{O}(\mathscr{E}, b, \mathscr{O}_X),$$

called the *vector representation* of the Clifford group. There's an exact sequence of group schemes,

$$1 \to \mathbb{G}_{\mathrm{m}} \to \mathbf{\Gamma}(\mathscr{E}, b, \mathscr{O}_X) \to \mathbf{O}(\mathscr{E}, b, \mathscr{O}_X) \to 1, \tag{1.11}$$

in the étale topology on $X$. This may be checked for strictly henselian local rings, where $\mathbf{O}$ is then generated by hyperplane reflections (with the single exception of the hyperbolic plane of rank 4 in characteristic 2, which can be handled separately).

Restricting the vector representation to the pin group yields the *pinor sequence*,

$$1 \to \mu_2 \to \mathbf{Pin}(\mathscr{E}, b, \mathscr{O}_X) \to \mathbf{O}(\mathscr{E}, b, \mathscr{O}_X) \to 1, \tag{1.12}$$

which is an exact sequence of smooth group schemes in the étale topology on $X$.

### 1.5.2 The generalized Clifford algebra

Now let $(\mathscr{E}, b, \mathscr{L})$ be an $\mathscr{L}$-valued quadratic form over $X$. Let $L(\mathscr{L}) = \oplus_{n\in\mathbb{Z}}\mathscr{L}^{\otimes n}$ be the $\mathscr{O}_X$-sheaf of *Laurent algebras* or *Rees algebras* and define $\mathbb{V}^\bullet(\mathscr{L}) = \mathbf{Spec}\ L(\mathscr{L})$. Then $p : \mathbb{V}^\bullet(\mathscr{L}) \to X$ is identified with the total space of the line bundle $\mathscr{L}$ on $X$ with the zero section removed, and satisfies the following universal property. Now, given a morphism of schemes $p' : X' \to X$ such that $p'^*\mathscr{L} \cong \mathscr{O}_{X'}$, there exists a morphism of schemes $t : X' \to \mathbb{V}^\bullet(\mathscr{L})$ so that $p' = p \circ t$. In fact, there is a canonical identification $p^*\mathscr{L} = \mathscr{O}_{\mathbb{V}^\bullet(\mathscr{L})}$ defined using the multiplication in the Laurent algebra $L(\mathscr{L})$.

**Definition 1.36.** Let $(\mathscr{E}, b, \mathscr{L})$ be an $\mathscr{L}$-valued symmetric bilinear form over a scheme $X$ and $p : \mathbb{V}^\bullet(\mathscr{L}) \to X$ the punctured total space of the line bundle $\mathscr{L}$ over $X$. Then $p^*(\mathscr{E}, b, \mathscr{L})$ has a natural structure of ($\mathscr{O}_{\mathbb{V}^\bullet(\mathscr{L})}$-valued) symmetric bilinear form on $\mathbb{V}^\bullet(\mathscr{L})$ and define the *generalized Clifford algebra* of $(\mathscr{E}, b, \mathscr{L})$ as the quasi-coherent sheaf of $\mathscr{O}_X$-algebras,

$$\widetilde{\mathscr{C}}(\mathscr{E}, b, \mathscr{L}) = p_*\mathscr{C}(p^*(\mathscr{E}, b, \mathscr{L})),$$

where $\mathscr{C}$ is the classical Clifford algebra construction. Then $\widetilde{\mathscr{C}}(\mathscr{E}, b, \mathscr{L})$ is a $\mathbb{Z}/2\mathbb{Z}$-graded $L(\mathscr{L})$-algebra, and via the $\mathbb{Z}$-grading of $L(\mathscr{L})$ as an $\mathscr{O}_X$-algebra, $\widetilde{\mathscr{C}}(\mathscr{E}, b, \mathscr{L})$ is naturally a $\mathbb{Z}$-graded $\mathscr{O}_X$-algebra. We call the 0th graded piece the *even Clifford algebra* $\mathscr{C}_0(\mathscr{E}, b, \mathscr{L})$ and the 1st graded piece the *Clifford module* $\mathscr{C}_1(\mathscr{E}, b, \mathscr{L})$. We often simply write $\widetilde{\mathscr{C}}$, $\mathscr{C}_0$, and $\mathscr{C}_1$ for these $\mathscr{O}_X$-algebras.

The generalized Clifford algebra is functorial for pull backs, has a direct construction as a quotient of the tensor algebra, and satisfies a graded universal property, analogous to the classical Clifford algebra, see Bichsel/Knus [7, §3].

Locally, the even Clifford algebra and Clifford module have descriptions reminiscent of the Poincaré-Birkhoff-Witt theorem, see Bichsel/Knus [7, Proposition 3.5].

**Proposition 1.37.** *Let $(\mathscr{E}, b, \mathscr{L})$ be an $\mathscr{L}$-valued symmetric bilinear form of rank $n = 2m$ or $n = 2m + 1$. If $\mathscr{E}$ is free with basis $\{e_1, \dots, e_n\}$ and $\mathscr{L}$ free with basis $l$, then $\mathscr{C}_0$ is a free $\mathscr{O}_X$-algebra with basis,*

$$\{1, e_{i_1} \cdots e_{i_{2k}} \otimes l^{-k} : 1 \le k \le m, 1 \le i_1 < \cdots < i_{2k} \le n\},$$

*and $\mathscr{C}_1$ is a free $\mathscr{O}_X$-module with basis,*

$$\{e_{i_1} \cdots e_{i_{2k+1}} \otimes l^{-k} : 1 \le k \le m, 1 \le i_1 < \cdots < i_{2k+1} \le n\}.$$

As a corollary, we see that if $(\mathscr{E}, b, \mathscr{L})$ is a bilinear space of rank $n$ then $\mathscr{C}_0(\mathscr{E}, b, \mathscr{L})$ is a locally free $\mathscr{O}_X$-algebra of rank $2^{n-1}$, $\mathscr{C}_1(\mathscr{E}, b, \mathscr{L})$ is a locally free $\mathscr{O}_X$-module of rank $2^{n-1}$, and there's a canonical monomorphism $i : \mathscr{E} \to \mathscr{C}_1(\mathscr{E}, b, \mathscr{L})$.

#### Patching classical even Clifford algebras

Following Parimala and Sridharan [40, §4], we may also construct $\mathscr{C}_0(\mathscr{E}, b, \mathscr{L})$ by patching together classical even Clifford algebras on an open cover of $X$. Choose a Zariski affine open cover, $\mathscr{U} = \{U_i\}_{i\in I}$, of $X$ trivializing $\mathscr{L}$ by $\varphi_i : \mathscr{O}_{U_i} \xrightarrow{\sim} \mathscr{L}|_{U_i}$, and let $a_{ij} = {\varphi_{i\underline{j}}}^{-1}\varphi_{\underline{i}j} \in \mathbb{G}_{\mathrm{m}}(U_{ij})$, for $(i,j) \in I^2$, be the corresponding Čech 1-cocycle representing $\mathscr{L}$. Now, for each $i \in I$, the composition,

$$\varphi_i^{-1} \circ b|_{U_i} : \mathscr{E} \otimes \mathscr{E}|_{U_i} \to \mathscr{L}|_{U_i} \to \mathscr{O}_{U_i},$$

defines an $\mathscr{O}_{U_i}$-valued quadratic space on $U_i$, and that for each $(i,j) \in I^2$, one checks that the identity map on $\mathscr{E}|_{U_{ij}}$ defines a similarity transformation,

$$(\mathrm{id}|_{U_{ij}}, a_{ij}) : (\mathscr{E}|_{U_{ij}}, \varphi_j^{-1} \circ b|_{U_{ij}}, \mathscr{O}_{U_{ij}}) \to (\mathscr{E}|_{U_{ij}}, \varphi_i^{-1} \circ b|_{U_{ij}}, \mathscr{O}_{U_{ij}}).$$

Then by Proposition 1.35, the similarities lift to algebra isomorphisms of the (classical) even Clifford algebras,

$$\mathscr{C}_0(\mathrm{id}|_{U_{ij}}, a_{ij}) : \mathscr{C}_0(\mathscr{E}|_{U_{ij}}, \varphi_j^{-1} \circ b|_{U_{ij}}, \mathscr{O}_{U_{ij}}) \to \mathscr{C}_0(\mathscr{E}|_{U_{ij}}, \varphi_i^{-1} \circ b|_{U_{ij}}, \mathscr{O}_{U_{ij}}),$$

and form a Čech 1-cocycle by the diagram in Proposition 1.35 (since the multiplier coefficients form a Čech 1-cocycle). Thus by descent, these Clifford algebras patch to yield the $\mathscr{O}_X$-algebra $\mathscr{C}_0(\mathscr{E}, b, \mathscr{L})$ on $X$.

A similar patching construction can be given for $\mathscr{C}_1(\mathscr{E}, b, \mathscr{L})$ again using Proposition 1.35 to patch the local $\mathscr{C}_0$-bimodule structure (which is well-behaved with respect to similitudes) on $\mathscr{C}_1$, to yield the global $\mathscr{C}_0(\mathscr{E}, b, \mathscr{L})$-bimodule structure on $\mathscr{C}_1(\mathscr{E}, b, \mathscr{L})$.

In particular, if $\mathscr{L}$ is the trivial line bundle and we choose a trivialization $\varphi : \mathscr{O}_X \xrightarrow{\sim} \mathscr{L}$, then the similarity transformation $(\mathrm{id}, \varphi) : (\mathscr{E}, \varphi^{-1} \circ b, \mathscr{O}_X) \to (\mathscr{E}, b, \mathscr{L})$, induces an $\mathscr{O}_X$-algebra isomorphism of the classical and generalized even Clifford algebras $\mathscr{C}_0(\mathscr{E}, \varphi^{-1} \circ b, \mathscr{O}_X) \xrightarrow{\sim} \mathscr{C}_0(\mathscr{E}, b, \mathscr{L})$ and the Clifford bimodules. More generally, the generalized even Clifford algebra of $(\mathscr{E}, b, \mathscr{L})$ is isomorphic to a classical even Clifford if $\mathscr{L}$ is a square in the Picard group, and then any choice of scaling of $(\mathscr{E}, b, \mathscr{L})$ yields such an isomorphism, see Knus [34, IV §7.1.2].

### Structure of the center

The center $\widetilde{\mathscr{Z}}$ of the generalized Clifford algebra $\widetilde{\mathscr{C}}(E, b, \mathscr{L})$ is a $\mathbb{Z}$-graded $\mathscr{O}_X$-algebra, with degree 0 and 1 parts denoted by $\mathscr{Z}_0$ and $\mathscr{Z}_1$, respectively. In analogy with Proposition 1.32, we have the following, see Bichsel/Knus [7, Theorem 3.7].

**Proposition 1.38.** *Let $(\mathscr{E}, b, \mathscr{L})$ be an $\mathscr{L}$-valued bilinear space of rank $n$ on $X$.*

*a) If $n$ is even, then $\mathscr{Z}_0 = \mathscr{O}_X \cdot 1_{\mathscr{C}_0}$ and $\mathscr{Z}_1 = 0$. If $\mathscr{Z}$ is the center of the even Clifford algebra $\mathscr{C}_0$, then $\mathscr{Z}$ is a rank 2 étale $\mathscr{O}_X$-algebra and $\mathscr{C}_0$ is an Azumaya algebra over $\mathbf{Spec}\, \mathscr{Z}$.*

*b) If $n$ is odd, then $\mathscr{C}_0$ is an Azumaya algebra over $X$. Finally, $\mathscr{Z}_0 = \mathscr{O}_X \cdot 1_{\mathscr{C}}$ and the multiplication in the generalized Clifford algebra induces an $\mathscr{L}$-valued line $\mathscr{Z}_1 \otimes \mathscr{Z}_1 \xrightarrow{\sim} \mathscr{L}$ on the line bundle $\mathscr{Z}_1$.*

If $\mathscr{L}$ is a square, then the classical and generalized notions of even Clifford algebra coincide.

### The even Clifford group

All algebra automorphisms and antiautomorphisms of the classical Clifford algebra descend to the generalized Clifford algebra and its degree 0 part. The canonical antiautomorphism on the tensor algebra of $E$ induces an $L(\mathscr{L})$-algebra antiautomorphism on the generalized Clifford algebra and thus an $\mathscr{O}_X$-algebra antiautomorphism on its degree 0 part, denoted by $\sigma$ and called the *canonical involution*.

**Definition 1.39.** We define the *even Clifford group* $\mathbf{S\Gamma}(\mathscr{E}, b, \mathscr{L})$ as the sheaf of groups associated to the presheaf

$$U \to \{x \in \mathscr{C}_0(\mathscr{E}, b, \mathscr{L})(U)^\times : x\mathscr{E}(U)x^{-1} \subset \mathscr{E}(U)\}$$

where $\mathscr{E} \hookrightarrow \mathscr{C}_1(\mathscr{E}, b, \mathscr{L})$ is the canonical inclusion.

**Proposition 1.40.** *Let $(\mathscr{E}, b, \mathscr{L})$ be a symmetric bilinear space on $X$.*

*a) The even Clifford group is represented by a smooth affine algebraic group scheme in the étale topology on $X$.*

*b) The* vector representation *induces an exact sequence,*

$$1 \to \mathbb{G}_{\mathrm{m}} \to \mathbf{S\Gamma}(\mathscr{E}, b, \mathscr{L}) \xrightarrow{r} \mathbf{SO}(\mathscr{E}, b, \mathscr{L}) \to 1,$$

*in the étale topology on $X$.*

*c) The restriction of the Clifford norm to the even Clifford group defines the spin group via the exact sequence,*

$$1 \to \mathbf{Spin}(\mathscr{E}, b, \mathscr{L}) \to \mathbf{S\Gamma}(\mathscr{E}, b, \mathscr{L}) \xrightarrow{N} \mathbb{G}_{\mathrm{m}} \to 1,$$

*in the étale topology on $X$. The spin group is a smooth affine algebraic group scheme over $X$.*

*Proof.* As for *a)*, the smoothness and representability will follow from *b)*, which is reduced, in view of §1.5.2 (since the exactness is a local question), to the exactness of the corresponding classical vector representation §1.5.1. Similarly for *c)*, with the smoothness and representability being a consequence of the exactness of the sequence, which is reduced to the classical case. □

As in the classical $\mathscr{O}_X$-valued case, restricting the vector representation of the Clifford group to the spin group yields a double cover of the special orthogonal group. The proof proceeds exactly as in the classical case, that is, by proving the surjectivity of the spin cover locally in the étale topology at geometric points using the standard generation of the orthogonal group by hyperplane reflections.

**Proposition 1.41.** *Let $(\mathscr{E}, b, \mathscr{L})$ be an $\mathscr{L}$-valued quadratic space, then there's a commutative diagram,*

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
1 \arrow[r] & \boldsymbol{\mu}_2 \arrow[r] \arrow[d] & \mathbf{Spin}(\mathscr{E}, b, \mathscr{L}) \arrow[r, "r"] \arrow[d] & \mathbf{SO}(\mathscr{E}, b, \mathscr{L}) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathbb{G}_{\mathrm{m}} \arrow[r] \arrow[d, "2"] & \mathbf{S\Gamma}(\mathscr{E}, b, \mathscr{L}) \arrow[r, "r"] \arrow[d, "N"] & \mathbf{SO}(\mathscr{E}, b, \mathscr{L}) \arrow[r] & 1 \\
 & \mathbb{G}_{\mathrm{m}} \arrow[r, equal] \arrow[d] & \mathbb{G}_{\mathrm{m}} \arrow[d] & & \\
 & 1 & 1 & &
\end{tikzcd}$$

*of group schemes in the étale topology on $X$ with exact rows and columns.*

# Chapter 2

# Cohomological invariants of line bundle-valued forms

In attempting to construct a cohomological invariant for $\mathscr{L}$-valued symmetric bilinear forms that generalizes the 2nd Hasse-Witt invariant, a natural approach is to construct a pinor sequence for the orthogonal group of an $\mathscr{L}$-valued form, and then interpret the second coboundary map in nonabelian étale Čech cohomology as a map from isometry classes of $\mathscr{L}$-valued symmetric bilinear forms to $H^2_{\text{ét}}(X, \boldsymbol{\mu}_2)$. Two complications arise for this approach.

First, already in our study of the spinor sequence for $\mathscr{L}$-valued forms in §1.5.2, we encountered the fact that in general there's only a canonically defined *even* Clifford group (not a full Clifford group); thus there exists only a spinor sequence and not a general pinor sequence. In particular, one cannot hope to define a Hasse-Witt invariant for all $\mathscr{L}$-valued forms simultaneously, but only for those of a given fixed discriminant. This is a new feature of the theory of $\mathscr{L}$-valued symmetric bilinear forms (especially when $\mathscr{L}$ is not a square in the Picard group).

The second complication is perhaps a matter of taste. Taking the second coboundary map of the spinor sequence for the special orthogonal group of an $\mathscr{L}$-valued form as the definition of a cohomological invariant is one thing, computing it is quite another. The issue at stake is that the special orthogonal group scheme of an $\mathscr{L}$-valued form (when $\mathscr{L}$ is not a square in the Picard group) does not come from base change of any group scheme over $\operatorname{Spec} \mathbb{Z}[\frac{1}{2}]$. It lives properly on $X$. The corresponding spinor sequence arises as a twist of the classical spinor sequence by a cocycle in $\mathbf{PO}$ that cannot be lifted to a cocycle in $\mathbf{O}$, i.e. a so-called *outer twist* or *outer form*. Serre warns [47, I, §5.5, Remarque] that the category of torsors for a group has in general no relation to the category of torsors of an outer form of the group. Already, we see quite plainly that the categories of bilinear forms of odd rank with values in $\mathscr{O}_X$ and with values in a non-square line bundle $\mathscr{L}$ couldn't be more disparate: one is quite rich while the other is empty. This is the obstruction that we overcome below.

Considering torsors for an outer twisted orthogonal group is equivalent to considering "twisted torsors" for a standard (untwisted) orthogonal group. The nice feature is that our twisted torsors are twisted by a cocycle with values in a standard (untwisted) orthogonal similitude group. We then realize that the coboundary map of the twisted spinor sequence is strongly related to the coboundary maps for a new Clifford sequence involving a four-fold covering of the orthogonal similitude groups. In general, this new coboundary map has independent interest as a similarity class cohomological invariant for $\mathscr{L}$-valued symmetric bilinear forms with fixed discriminant, and lives in an étale cohomology group with coefficients "modulo 4." In turn, this coboundary map for the orthogonal similitude group can be computed explicitly for many general families of $\mathscr{L}$-valued forms. It is perhaps a round-about path to the construction of cohomological invariants for $\mathscr{L}$-valued symmetric bilinear forms, but the only feasible way in light of the important difficulties presented by the $\mathscr{L}$-valued case.

## 2.1 A similarity class cohomological invariant

We begin by developing the necessary tools to construct the cohomological coboundary map of a new Clifford sequence involving the orthogonal similitude group. As always, let $X$ be a noetherian separated scheme with $\frac{1}{2} \in \mathscr{O}_X$ and considered always in the étale topology. In the interest of simplifying the exposition, we will assume that $X$ is connected. Throughout, we will fix an $\mathscr{O}_X$-valued symmetric bilinear space $\mathscr{H} = (\mathscr{H}, h, \mathscr{O}_X)$ of rank $n$ on $X$, and unless explicitly stated otherwise, we will omit the dependence on this space in the notation for the classical (special) orthogonal, (s)pin, (proper) similitude, and (even) Clifford group schemes. When denoting the isometry class of an $\mathscr{O}_X$-valued symmetric bilinear space, often we will write $(\mathscr{H}, h)$ instead of $(\mathscr{H}, h, \mathscr{O}_X)$.

### 2.1.1 Classical cohomological invariants

#### Chern classes

Recall Grothendieck's construction[24] of *Chern classes modulo $l$* in étale cohomology, for $l \geq 2$ such that $\frac{1}{l} \in \mathscr{O}_X$,

$$c_i(\mathscr{E}, \boldsymbol{\mu}_l) \in H^{2i}_{\text{ét}}(X, \boldsymbol{\mu}_l^{\otimes i}),$$

of a vector bundle $\mathscr{E}$ on $X$: if $\mathscr{E}$ a line bundle, define $c_1(\mathscr{E}, \boldsymbol{\mu}_l) \in H^2_{\text{ét}}(X, \boldsymbol{\mu}_l)$ as the image,

$$\begin{array}{rcl} H^1_{\text{ét}}(X, \mathbb{G}_{\mathrm{m}}) & \to & H^2_{\text{ét}}(X, \boldsymbol{\mu}_l) \\ [\mathscr{E}] & \mapsto & c_1(\mathscr{E}, \boldsymbol{\mu}_l) \end{array}$$

of the isomorphism class, $[E] \in \operatorname{Pic}(X) \xrightarrow{\sim} H^1_{\text{ét}}(X, \mathbb{G}_{\mathrm{m}})$, under the first coboundary map arising from the Kummer sequence,

$$1 \to \boldsymbol{\mu}_l \to \mathbb{G}_{\mathrm{m}} \xrightarrow{l} \mathbb{G}_{\mathrm{m}} \to 1,$$

which is exact in the étale topology on $X$. For a vector bundle of general rank, use the splitting principle for Chern classes. In particular, for any vector bundle $\mathscr{E}$,

$$c_1(\mathscr{E}, \boldsymbol{\mu}_l) = c_1(\det \mathscr{E}, \boldsymbol{\mu}_l) \in H^2_{\text{ét}}(X, \mu_l).$$

#### Classical 1st Hasse-Witt invariant

Recall the classical torsor-theoretic construction of the $(\mathscr{H}, h)$-base pointed *1st Hasse-Witt invariant*,

$$w_1^{\mathscr{H}}(\mathscr{E}, b) \in H^1_{\text{ét}}(X, \boldsymbol{\mu}_2),$$

of an $\mathscr{O}_X$-valued symmetric bilinear space $(\mathscr{E}, b)$ on $X$: it's the image,

$$\begin{array}{rcl} H^1_{\text{ét}}(X, \mathbf{O}) & \to & H^1_{\text{ét}}(X, \boldsymbol{\mu}_2) \\ [\mathscr{E}, b] & \mapsto & w_1^{\mathscr{H}}(\mathscr{E}, b) \end{array}$$

of the isometry class $[\mathscr{E}, b] \in H^1_{\text{ét}}(X, \mathbf{O})$, under the map on cohomology induced by the *determinant sequence*,

$$1 \to \mathbf{SO} \to \mathbf{O} \xrightarrow{\det} \boldsymbol{\mu}_2 \to 1,$$

for the orthogonal group scheme, see Remark 1.12. For example, if $(\mathscr{H}, h)$ is the standard sum-of-squares form of rank $n$, then we write $\mathbf{O}_n = \mathbf{O}(\mathscr{H}, h)$ and $w_1(\mathscr{E}, b) = w_1^{\mathscr{H}}(\mathscr{E}, b)$. By §1.2.1, $w_1(\mathscr{E}, b) \in H^1_{\text{ét}}(X, \boldsymbol{\mu}_2)$ coincides the isometry class of the determinant form. If $(\mathscr{H}, h)$ is the hyperbolic form of rank $n = 2m$ with trivial lagrangian, then we write $\mathbf{O}_{m,m} = \mathbf{O}(\mathscr{H}, h)$. In this case, $w_1^{\mathscr{H}}(\mathscr{E}, b) \in H^1_{\text{ét}}(X, \boldsymbol{\mu}_2)$ coincides with the classical *signed discriminant*.

**Classical 2nd Hasse-Witt invariant**

Recall the classical *vector representation* of the Clifford group scheme,

$$\begin{array}{rcccl} 1 \to \mathbb{G}_{\mathrm{m}} & \to & \mathbf{\Gamma} & \xrightarrow{r} & \mathbf{O} \to 1 \\ & & x & \mapsto & v \mapsto x\, v\, I(x)^{-1} \end{array} \tag{2.1}$$

where $I$ is the *standard automorphism* of the Clifford algebra (i.e. the $\mathscr{O}_X$-algebra automorphism induced by negation on $\mathscr{H}$), see §1.5.1. Also recall the *Clifford norm* sequence defining the pin group,

$$\begin{array}{rcccl} 1 \to \mathbf{Pin} & \to & \mathbf{\Gamma} & \xrightarrow{N} & \mathbb{G}_{\mathrm{m}} \to 1 \\ & & x & \mapsto & x\, \sigma(x) \end{array} \tag{2.2}$$

where $\sigma$ is the *canonical involution* of the Clifford algebra (i.e. the $\mathscr{O}_X$-algebra antiautomorphism induced by the identity on $H$).

Finally, recall the classical torsor theoretic construction of the $(\mathscr{H}, h, \mathscr{O}_X)$-base pointed *2nd Hasse-Witt invariant*,

$$w_2^{\mathscr{H}}(\mathscr{E}, b) \in H^2_{\text{ét}}(X, \boldsymbol{\mu}_2),$$

of an $\mathscr{O}_X$-valued symmetric bilinear space $(\mathscr{E}, b)$ on $X$: it's the image,

$$\begin{array}{rcl} H^1_{\text{ét}}(X, \mathbf{O}) & \to & H^2_{\text{ét}}(X, \boldsymbol{\mu}_2) \\ [\mathscr{E}, b] & \mapsto & w_2^{\mathscr{H}}(\mathscr{E}, b) \end{array}$$

of the isometry class $[\mathscr{E}, b] \in H^1_{\text{ét}}(X, \mathbf{O})$, under the first coboundary map arising from the *pinor sequence*,

$$1 \to \boldsymbol{\mu}_2 \to \mathbf{Pin} \to \mathbf{O} \to 1, \tag{2.3}$$

for the orthogonal group scheme, see §1.5.1. For example, if $(\mathscr{H}, h)$ is the standard sum-of-squares form, then $w_2(\mathscr{E}, b) = w_2^{\mathscr{H}}(\mathscr{E}, b)$ is the classical Hasse-Witt invariant. If $(\mathscr{H}, h)$ is the hyperbolic form with trivial lagrangian, then $e_2(\mathscr{E}, b) = w_2^{\mathscr{H}}(\mathscr{E}, b) \in H^2_{\text{ét}}(X, \boldsymbol{\mu}_2)$ coincides with the *Clifford invariant*.

**Comparing base forms**

Let $a = (a_{ij})$ be an étale Čech 1-cocycle (for some cover $\mathscr{U}$ of $X$) representing the class of $(\mathscr{E}, b)$ in $H^1_{\text{ét}}(X, \mathbf{O})$. In the literature $w_i^{\mathscr{H}}(\mathscr{E}, b)$ is often thought of as the standard sum-of-squares base pointed $i$th Hasse-Witt invariant $w_i(\mathscr{H}_a)$, of the form $\mathscr{H} = (\mathscr{H}, b)$ twisted by the cocycle $a$. Serre [47, Annexe §2.2] and Cassou-Noguès/Erez/Taylor [12] denote $w_i^{\mathscr{H}}$ by $\delta^i = \delta^i_{\mathscr{H}}$. We prefer to guard, in the notation, the reliance on the base form when discussing Hasse-Witt invariants. Indeed, for $\mathscr{L}$-valued forms there is no standard choice of base form possible. That being said, comparison of Hasse-Witt invariants for different base points is possible, see Cassou-Noguès/Erez/Taylor [12, Theorem 0.2].

**Proposition 2.1.** *Let $(\mathscr{H}, h)$, $(\mathscr{H}', h')$, and $(\mathscr{E}, b)$ be $\mathscr{O}_X$-valued symmetric bilinear spaces of rank $n$ on $X$. Then*

*a)* $w_1^{\mathscr{H}'}(\mathscr{E}, b) = w_1^{\mathscr{H}'}(\mathscr{H}, h) + w_1^{\mathscr{H}}(\mathscr{E}, b)$

*b)* $w_2^{\mathscr{H}'}(\mathscr{E}, b) = w_2^{\mathscr{H}'}(\mathscr{H}, h) + w_1^{\mathscr{H}'}(\mathscr{H}, h) \cdot w_1^{\mathscr{H}}(\mathscr{E}, b) + w_2^{\mathscr{H}}(\mathscr{E}, b)$

*in the étale cohomology groups $H^1_{\text{ét}}(X, \boldsymbol{\mu}_2)$ and $H^2_{\text{ét}}(X, \boldsymbol{\mu}_2)$, respectively.*

**Orthogonal splitting principle for Hasse-Witt invariants**

Analogous to the well-known splitting principle for Chern classes of vector bundles, there is an *orthogonal splitting principle* for Hasse-Witt invariants, hinted at by Grothendieck [24] and also implicit (in a weak version) in Fulton [19, §2] and Edidin/Graham [16, §7].

We say that a scheme $X$ has the *Krull-Schmidt property* if its exact category of vector bundles has the Krull-Schmidt property, i.e. every object is uniquely (up to reordering) a sum of indecomposable objects. For example, if $X$ is a proper variety over a field, then $X$ has the Krull-Schmidt property.

**Theorem 2.2** (Orthogonal splitting principle for Hasse-Witt invariants)**.** *Let $X$ be a scheme with the Krull-Schmidt property and with $\frac{1}{2} \in \mathscr{O}_X$. Let $(\mathscr{H}, h)$ and $\mathscr{O}_X$-valued symmetric bilinear space on $X$. Then there's a morphism of schemes $f : Y \to X$ satisfying*

- *$f^* : H^i_{\text{ét}}(X, \boldsymbol{\mu}_2) \to H^i_{\text{ét}}(Y, \boldsymbol{\mu}_2)$ is injective on étale cohomology modulo 2, and*
- *$f^*(\mathscr{H}, h)$ is isometric to an orthogonal sum of hyperbolic planes $H_{\mathscr{O}_X}(\mathscr{V}_i)$ for line bundles $\mathscr{V}_i$ and of $\mathscr{O}_X$-valued lines $(\mathscr{N}_j, n_j)$ for self-dual line bundles $\mathscr{N}_j$ on $Y$.*

*Proof.* By the splitting principle for exact sequences of vector bundles (see Theorem 1.29 and Fulton [19, §2]) and by the classical Witt cancellation principle in the affine case (see Knebusch [30, I §2-3]), the proof of which only depends on the orthogonal complements being direct summands, we can find a morphism $f : Y \to X$ which is an isomorphism on cohomology and such that $f^*(\mathscr{H}, h)$ is an orthogonal sum of a split metabolic space and an anisotropic space $\mathscr{M} \perp (\mathscr{H}_0, h_0)$ on $Y$. This is also possible by pulling back to an adapted "maximal isotropic flag variety", and generalizing results of in Fulton [19, §2]. We can use the splitting principle for metabolic spaces (see Theorem 1.29) to split $\mathscr{M}$ into an orthogonal sum of hyperbolic planes. We are left with anisotropic forms. By using the splitting principle for vector bundles, we can pull back further to split these into direct sums of line bundles, and then by Knus [34, II §6.3.1] (this is where we need the Krull-Schmidt property) these will decompose into orthogonal sums of self-dual line bundles and (possibly more) hyperbolic spaces. $\square$

As in the case of Chern classes, to verify a statement about Hasse-Witt invariants of symmetric bilinear spaces, it's sufficient to consider only spaces that are sums of hyperbolic planes and lines. The 1st Hasse-Witt invariant of an $\mathscr{O}_X$-valued line $w_1(\mathscr{N}, n)$ is the isomorphism class of $(\mathscr{N}, n)$ in $H^1_{\text{ét}}(X, \boldsymbol{\mu}_2)$. We also have a formula due to Esnault/Kahn/Viehweg [17, Proposition 5.5], for the Hasse-Witt invariants of metabolic spaces.

**Theorem 2.3.** *Let $(\mathscr{H}, h)$ be a metabolic plane with lagrangian $\mathscr{V}$. Then*

$$w(\mathscr{H}, h) = 1 + w_1(\mathscr{H}, h) + w_2(\mathscr{H}, h) = 1 + (-1) + c_1(\mathscr{V}, \boldsymbol{\mu}_2)$$

*in $H^0_{\text{ét}}(X, \boldsymbol{\mu}_2) \oplus H^1_{\text{ét}}(X, \boldsymbol{\mu}_2) \oplus H^2_{\text{ét}}(X, \boldsymbol{\mu}_2)$.*

Together with all this, the Whitney sum formula

$$w_2((\mathscr{H}, h) \perp (\mathscr{H}', h')) = w_2(\mathscr{H}, h) + w_1(\mathscr{H}, h) \cdot w_1(\mathscr{H}', h') + w_2(\mathscr{H}', h')$$

allows us to compute the 2nd Hasse-Witt invariants (the higher ones as well) of any symmetric bilinear space once we know a splitting.

### 2.1.2 The similarity 1st Hasse-Witt invariant

For even rank forms, the classical (unsigned) discriminant form generalizes to bilinear forms with values in line bundles, see §1.2.1 or Parimala/Sridharan [40, §4] for the analogue of the signed discriminant.

**Definition 2.4.** Let $(\mathscr{H}, h)$ be an $\mathscr{O}_X$-valued symmetric bilinear space of even rank $n$ on $X$. We define the $(\mathscr{H}, h)$-base pointed *similarity 1st Hasse-Witt invariant*,

$$gw_1^{\mathscr{H}}(\mathscr{E}, b, \mathscr{L}) \in H^1_{\text{ét}}(X, \boldsymbol{\mu}_2),$$

of an $\mathscr{L}$-valued symmetric bilinear space $(\mathscr{E}, b, \mathscr{L})$ of rank $n$ on $X$: it's the image,

$$\begin{array}{rcl} H^1_{\text{ét}}(X, \mathbf{GO}) & \to & H^1_{\text{ét}}(X, \boldsymbol{\mu}_2) \\ [\mathscr{E}, b, \mathscr{L}] & \mapsto & gw_1^{\mathscr{H}}(\mathscr{E}, b, \mathscr{L}) \end{array}$$

of the similarity class $[\mathscr{E}, b, \mathscr{L}] \in H^1_{\text{ét}}(X, \mathbf{GO})$, under the map on cohomology induced by the *discriminant sequence*

$$1 \to \mathbf{GSO} \to \mathbf{GO} \xrightarrow{\text{disc}} \boldsymbol{\mu}_2 \to 1,$$

for the orthogonal similitude group §1.2.2.

By Remark 1.12, the discriminant sequence for the orthogonal similitude group restricts to the determinant sequence for the orthogonal group, and so the similarity 1st Hasse-Witt invariant generalizes the classical 1st Hasse-Witt invariant. Due to our identification $\mathbf{O}(\text{disc}(\mathscr{H}, h)) \xrightarrow{\sim} \boldsymbol{\mu}_2$, we have the following equation

$$gw_1^{\mathscr{H}}(\mathscr{E}, b, \mathscr{L}) = \text{disc}(\mathscr{E}, b, \mathscr{L}) - \text{disc}(\mathscr{H}, h, \mathscr{O}_X)$$

interpreting isometry classes of $\mathscr{O}_X$-lines as elements of $H^1_{\text{ét}}(X, \boldsymbol{\mu}_2)$, see §1.2.2. In particular, $gw_1^{\mathscr{H}}(\mathscr{E}, b, \mathscr{L})$ vanishes if and only if there's an isometry $\text{disc}(\mathscr{H}, h, \mathscr{O}_X) \cong \text{disc}(\mathscr{E}, b, \mathscr{L})$ of discriminant forms.

### 2.1.3 A Clifford sequence for the orthogonal similitude group

To construct the required four-fold cover of the orthogonal similitude group by the Clifford group, we first pass through some intermediary exact sequences of group schemes and their implications on categories of torsors.

#### First intermediary exact sequence

Taking the cartesian product of the Clifford group homomorphisms (2.1) and (2.2) yields an exact sequence,

$$1 \to \boldsymbol{\mu}_2 \to \boldsymbol{\Gamma} \xrightarrow{N \times r} \mathbb{G}_{\mathrm{m}} \times \mathbf{O} \to 1, \tag{2.4}$$

in the étale topology on $X$, where $\boldsymbol{\mu}_2 \to \mathbb{G}_{\mathrm{m}} \to \boldsymbol{\Gamma}$ denotes the canonical inclusion of constants.

**Proposition 2.5.** *Exact sequence (2.4) gives rise to the coboundary map,*

$$\begin{array}{rcl} H^1_{\text{ét}}(X, \mathbb{G}_{\mathrm{m}} \times \mathbf{O}) \cong H^1_{\text{ét}}(X, \mathbb{G}_{\mathrm{m}}) \times H^1_{\text{ét}}(X, \mathbf{O}) & \to & H^2_{\text{ét}}(X, \boldsymbol{\mu}_2) \\ {[\mathscr{L}, (\mathscr{E}, b)]} & \mapsto & c_1(\mathscr{L}, \boldsymbol{\mu}_2) + w_2^{\mathscr{H}}(\mathscr{E}, b). \end{array}$$

*Proof.* One needs only to consider the following commutative diagrams of group schemes in the étale topology on $X$,

$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \boldsymbol{\mu}_2 \arrow[r] \arrow[d, equal] & \mathbb{G}_{\mathrm{m}} \arrow[r, "2"] \arrow[d] & \mathbb{G}_{\mathrm{m}} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \boldsymbol{\mu}_2 \arrow[r] & \boldsymbol{\Gamma} \arrow[r, "N \times r"] \arrow[d, "r"'] & \mathbb{G}_{\mathrm{m}} \times \mathbf{O} \arrow[r] \arrow[d] & 1 \\
 & & \mathbf{O} \arrow[r, equal] \arrow[d] & \mathbf{O} \arrow[d] & \\
 & & 1 & 1 &
\end{tikzcd} \tag{2.5}$$

$$
\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \boldsymbol{\mu}_2 \arrow[r] \arrow[d, equal] & \mathbf{Pin} \arrow[r, "r"] \arrow[d] & \mathbf{O} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \boldsymbol{\mu}_2 \arrow[r] & \mathbf{\Gamma} \arrow[r, "N \times r"] \arrow[d, "N"'] & \mathbb{G}_{\mathrm{m}} \times \mathbf{O} \arrow[r] \arrow[d] & 1 \\
 & & \mathbb{G}_{\mathrm{m}} \arrow[r, equal] \arrow[d] & \mathbb{G}_{\mathrm{m}} \arrow[d] & \\
 & & 1 & 1 &
\end{tikzcd}
\tag{2.6}
$$

and their implications on étale cohomology groups. Indeed, from the commutativity of the first diagram, $[\mathscr{L}, (\mathscr{H}, h)] \mapsto c_1(\mathscr{L}, \boldsymbol{\mu}_2)$ under the coboundary map, and from he second diagram, $[\mathscr{O}_X, (\mathscr{E}, b)] \mapsto w_2^{\mathscr{H}}(\mathscr{E}, b)$. Finally, the action of $H^1_{\text{ét}}(X, \mathbb{G}_{\mathrm{m}})$ on $H^1_{\text{ét}}(X, \mathbb{G}_{\mathrm{m}} \times \mathbf{O})$ in the first diagram is given by

$$[\mathscr{L}_1]\,[\mathscr{L}_2, (\mathscr{E}, b, )] = [\mathscr{L}_1 \otimes \mathscr{L}_2, (\mathscr{E}, b)],$$

so that

$$[\mathscr{L}, (\mathscr{E}, b)] = [\mathscr{L}]\,[\mathscr{O}_X, (\mathscr{E}, b)] = [\mathscr{L}, (\mathscr{H}, h)]\,[\mathscr{O}_X, (\mathscr{E}, b)] \mapsto c_1(\mathscr{L}, \boldsymbol{\mu}_2) + w_2^{\mathscr{H}}(\mathscr{E}, b)$$

gives the formula. $\square$

**Second intermediary exact sequence**

There's a canonical scalar multiplication homomorphism

$$
\begin{array}{rcl}
\mathbb{G}_{\mathrm{m}} \times \mathbf{O} & \xrightarrow{m} & \mathbf{GO} \\
(l, \varphi) & \longmapsto & (l\varphi, l^2)
\end{array}
$$

which is an epimorphism in the étale topology on $X$. Indeed, for each section $(\psi, \mu_\psi)$ of $\mathbf{GO}$ over $U \to X$, we can find an étale cover of $U$ where the multiplying coefficient $\mu_\psi$ has a square root $\tilde{\mu}_\psi$. Then $(\tilde{\mu}_\psi, \tilde{\mu}_\psi^{-1}\varphi) \mapsto (\psi, \mu_\psi)$ under the scalar multiplication homomorphism so that $m$ is locally (hence globally) an epimorphism in the étale topology on $X$. Thus there's an exact sequence,

$$1 \to \boldsymbol{\mu}_2 \to \mathbb{G}_{\mathrm{m}} \times \mathbf{O} \to \mathbf{GO} \to 1, \tag{2.7}$$

in the étale topology on $X$, where $\boldsymbol{\mu}_2 \to \mathbb{G}_{\mathrm{m}} \times \mathbf{O}$ is the diagonal inclusion.

**Proposition 2.6.** *Exact sequence (2.7) gives rise to the coboundary map,*

$$
\begin{array}{rcl}
H^1_{\text{ét}}(X, \mathbf{GO}) & \to & H^2_{\text{ét}}(X, \boldsymbol{\mu}_2) \\
[\mathscr{E}, b, \mathscr{L}] & \to & c_1(\mathscr{L}, \boldsymbol{\mu}_2).
\end{array}
$$

*Proof.* One needs only to consider the implications on étale cohomology of the following commutative diagram of groups schemes,

$$
\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
 & & \mathbf{O} \arrow[r, equal] \arrow[d] & \mathbf{O} \arrow[d] & \\
1 \arrow[r] & \boldsymbol{\mu}_2 \arrow[r] \arrow[d, equal] & \mathbb{G}_{\mathrm{m}} \times \mathbf{O} \arrow[r, "m"] \arrow[d] & \mathbf{GO} \arrow[r] \arrow[d, "\mu"] & 1 \\
1 \arrow[r] & \boldsymbol{\mu}_2 \arrow[r] & \mathbb{G}_{\mathrm{m}} \arrow[r, "2"] \arrow[d] & \mathbb{G}_{\mathrm{m}} \arrow[r] \arrow[d] & 1 \\
 & & 1 & 1 &
\end{tikzcd}
\tag{2.8}
$$

in the étale topology on $X$. Indeed, since $(\mathscr{H}, h)$ is an $\mathscr{O}_X$-valued space, there's a canonical identification $\mathbb{G}_{\mathrm{m}} = \mathbf{GL}(\mathscr{O}_X)$ (compare with Remark 1.6). The composition of maps on étale cohomology

$$
\begin{array}{ccccc}
H^1_{\text{ét}}(X, \mathbf{GO}) & \to & H^1_{\text{ét}}(X, \mathbb{G}_{\mathrm{m}}) & \to & H^2_{\text{ét}}(X, \boldsymbol{\mu}_2) \\
[\mathscr{E}, b, \mathscr{L}] & \mapsto & \mathscr{L} & \mapsto & c_1(\mathscr{L}, \boldsymbol{\mu}_2).
\end{array}
$$

yields the stated formula. □

**The kernel of the four-fold cover**

The composite homomorphism

$$
s = m \circ (N \times r) : \mathbf{\Gamma} \to \mathbb{G}_{\mathrm{m}} \times \mathbf{O} \to \mathbf{GO}, \tag{2.9}
$$

is an epimorphism in the étale topology on $X$.

**Definition 2.7.** Let $(\mathscr{H}, h)$ be an $\mathscr{O}_X$-valued symmetric bilinear space on $X$. Define $\boldsymbol{\kappa} = \boldsymbol{\kappa}(\mathscr{H}, h)$ to be the sheaf kernel of the homomorphism $s : \mathbf{\Gamma} \to \mathbf{GO}$.

By construction, $\boldsymbol{\kappa}$ is an order 4 subgroup scheme of the Clifford group and fits into a *funda-*

*mental diagram*

$$
\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \boldsymbol{\mu}_2 \arrow[r] \arrow[d, "\iota"] & \mathbf{Pin} \arrow[r, "r"] \arrow[d] & \mathbf{O} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \boldsymbol{\kappa} \arrow[r] \arrow[d] & \mathbf{\Gamma} \arrow[r, "s"] \arrow[d, "N"] & \mathbf{GO} \arrow[r] \arrow[d, "\mu"] & 1 \\
1 \arrow[r] & \boldsymbol{\mu}_2 \arrow[r] \arrow[d] & \mathbb{G}_{\mathrm{m}} \arrow[r, "2"] \arrow[d] & \mathbb{G}_{\mathrm{m}} \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}
\tag{2.10}
$$

of group schemes with exact rows and columns in the étale topology on $X$. Indeed, by the exactness of the pinor sequence (2.3), the intersection of the kernels of $N$ and $r$ is just the group $\{\pm 1\}$. The commutativity is a simple verification. The diagram shows that the middle row sequence defining $\boldsymbol{\kappa}$ is an "extension" of the Kummer sequence (the bottom row) by the pinor sequence (the top row).

We'll now calculate the local structure of the étale group scheme $\boldsymbol{\kappa}$, i.e. the isomorphism class of $\boldsymbol{\kappa}$ at any geometric point.

**Proposition 2.8.** *Let $(\mathscr{H}, h)$ be an $\mathscr{O}_X$-valued symmetric bilinear space of rank $n$ on $X$ and $\boldsymbol{\kappa} = \boldsymbol{\kappa}(\mathscr{H}, h)$ as defined above. Then in the étale topology on $X$, $\boldsymbol{\kappa}$ is locally isomorphic to*

$$
\begin{array}{ll}
\mathbb{Z}/4\mathbb{Z} & \textit{if } n \equiv 0, 1 \bmod 4 \\
\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \textit{if } n \equiv 2, 3 \bmod 4.
\end{array}
$$

*Proof.* The proof proceeds by a local Clifford group calculation. Locally in the étale topology, $(\mathscr{H}, h)$ is isometric to the standard sum-of-squares form, see Knus [34, IV §2.2.1, 3.2.1], with orthonormal basis $e_1, \dots, e_n$. Locally, we can also choose a square root of $-1$. By the fundamental diagram (2.10), $\boldsymbol{\kappa}$ is locally the group $\{\pm 1, \pm\epsilon\}$ for some local section $\epsilon$ of the Clifford algebra that satisfies $N(\epsilon) = -1$ and $r(\epsilon) = -\mathrm{id}$.

Consider the element $\epsilon = \sqrt{-1}\, e_1 \cdots e_n$ of the Clifford algebra. Recalling that

$$
e_i\, e_j = \left\{ \begin{array}{cl} 1 & \text{if } i = j \\ -e_j\, e_i & \text{if } i \neq j \end{array} \right. ,
$$

we compute

$$
N(\epsilon) = \epsilon\, \sigma(\epsilon) = \sqrt{-1}\, e_1 \cdots e_n \sqrt{-1}\, e_n \cdots e_1 = -1.
$$

In particular $\epsilon^{-1} = -\sigma(\epsilon) = -(-1)^{n(n-1)/2}\, \epsilon$. Also note that

$$
r(\epsilon)(e_i) = \epsilon\, e_i\, I(\epsilon)^{-1} = (-1)^{n-1}(-1)^n\, e_i \cdot \epsilon \cdot \epsilon^{-1} = -e_i,
$$

hence $r(\epsilon) = -\mathrm{id}$. Thus $-1$ and $\epsilon$ locally generate $\boldsymbol{\kappa}$ and we only need to calculate $\epsilon^2$ to know the group structure. We find that

$$
\epsilon^2 = -(-1)^{n(n-1)/2},
$$

so that locally, $\boldsymbol{\kappa}$ is as claimed. $\square$

*Remark* 2.9. Proposition 2.8 depends on our convention for the Clifford norm $N$. We take the convention set forth in Fröhlich [18, Appendix I]. The other convention, taken by Knus [34, IV §6.1] for instance, defines $N$ via the *standard involution* $\tau$ (i.e. the Clifford algebra antiautomorphism induced by negation on $\mathscr{H}$) and yields a different pin group (but the same spin group) and a different structure for $\boldsymbol{\kappa}$ in odd ranks.

By the fundamental diagram (2.10), the group scheme $\boldsymbol{\kappa}$ is an extension of $\boldsymbol{\mu}_2$ by $\boldsymbol{\mu}_2$,

$$1 \to \boldsymbol{\mu}_2 \xrightarrow{\iota} \boldsymbol{\kappa} \to \boldsymbol{\mu}_2 \to 1. \tag{2.11}$$

We now give a precise formula for the isomorphism class of the étale group scheme $\boldsymbol{\kappa}$. To this end, we use Proposition 2.8 to introduce a cohomology class that classifies $\boldsymbol{\kappa}$.

- For $n \equiv 2, 3 \bmod 4$, $\boldsymbol{\kappa}$ is a form of $\mathbb{Z}/4\mathbb{Z}$ and hence is classified by an element of $H^1_{\text{ét}}(X, \mathbf{Aut}(\mathbb{Z}/4\mathbb{Z})) \cong H^1_{\text{ét}}(X, \boldsymbol{\mu}_2)$.
- For $n \equiv 0, 1 \bmod 4$, $\boldsymbol{\kappa}$ is a form of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and hence is classified by an element of $H^1_{\text{ét}}(X, \mathbf{Aut}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}))$. While $\mathbf{Aut}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}))$ is isomorphic to the constant symmetric group $S_3$, acting on $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ by permuting the three subgroups of order 2, we claim that the image of $\iota$ is fixed by any cocycle representing $\boldsymbol{\kappa}$. Indeed, by the commutativity of the fundamental diagram (2.10), any cocycle representing the class of $\boldsymbol{\kappa}$ is given by conjugation by an element of the Clifford group of norm $-1$, which fixes the central subgroup scheme $\boldsymbol{\mu}_2 \to \boldsymbol{\kappa} \to \boldsymbol{\Gamma}$. Thus $\boldsymbol{\kappa}$ admits a "reduction of structure group" from $S_3$ to $S_2 \cong \mathbb{Z}/2\mathbb{Z} \cong \boldsymbol{\mu}_2$.

Thus in all cases, $\boldsymbol{\kappa} = \boldsymbol{\kappa}(\mathscr{H}, h)$ is classified up to isomorphism by an element

$$[\boldsymbol{\kappa}] \in H^1_{\text{ét}}(X, \boldsymbol{\mu}_2), \tag{2.12}$$

which we determine presently in terms of the 1st Hasse-Witt invariant of $(\mathscr{H}, h)$. First, we'll explicitly see what happens in the hyperbolic case.

**Example 2.10.** Let $(\mathscr{H}, h) = H_{\mathscr{O}_X}(\mathscr{O}_X^m)$ be a hyperbolic space on $X$ with trivial lagrangian, see §1.4.3, and $e_1, \dots, e_m, f_1, \dots, f_m$ be a choice of global sections forming a hyperbolic basis, i.e. $h(f_l, e_k) = \delta_{lk}$.

For $m$ odd, define a global section

$$\epsilon = \prod_{l=1}^{m} (1 - e_l f_l) \in \boldsymbol{\Gamma}(X),$$

of the Clifford group. Note that

$$(1 - e_l f_l) e_k = e_k - e_l(2h(f_l, e_k) - e_k f_l) = \begin{cases} e_k(1 - e_l f_l) & \text{if } k \neq l \\ -e_k(1 - e_l f_l) & \text{if } k = l \end{cases}$$

so that $r(\epsilon) = -\mathrm{id} \in \mathbf{O}(X)$. Also note that

$$N(1 - e_l f_l) = (1 - e_l f_l)(1 - f_l e_l) = 1 - (e_l f_l + f_l e_l) = -1,$$

so that $N(\epsilon) = (-1)^m = -1 \in \mathbb{G}_{\mathrm{m}}(X)$. Thus $\epsilon \in \boldsymbol{\kappa}(X)$ is a global section generating $\boldsymbol{\kappa}$ together with $-1$. Finally, since $\sigma(1 - e_l f_l) = -(1 - e_l f_l)$, we see that

$$\epsilon^2 = (-1)^m N(\epsilon) = -(-1)^m = 1,$$

so there's an induced isomorphism

$$\begin{array}{rcl} \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 & \xrightarrow{\sim} & \boldsymbol{\kappa} \\ \pm(1,1) & \mapsto & \pm 1 \\ \pm(-1,1) & \mapsto & \pm\epsilon \end{array}$$

of group schemes in the étale topology on $X$. This isomorphism clearly depends on our particular choice of $\epsilon$.

For $m$ even, consider the section

$$\epsilon = \prod_{l=1}^{m} (1 - e_l f_l) \otimes \sqrt{-1} \in \mathbf{\Gamma}(\widetilde{X}),$$

of the Clifford group over the étale double cover $\widetilde{X} = X \times_{\operatorname{Spec}\mathbb{Z}[\frac{1}{2}]} \operatorname{Spec}\mathbb{Z}[\frac{1}{2}, \sqrt{-1}] \to X$. Then as above, $r(\epsilon) = -\mathrm{id} \in \mathbf{O}(\widetilde{X})$. However, now $N(\epsilon) = -(-1)^m = -1 \in \mathbb{G}_{\mathrm{m}}(\widetilde{X})$ and $\epsilon^2 = -1$. Thus $\epsilon \in \boldsymbol{\kappa}(\widetilde{X})$ is a section over $\widetilde{X}$, and there's an isomorphism

$$\begin{array}{rcl} \mathbb{Z}/4\mathbb{Z} \cong \boldsymbol{\mu}_4|_{\widetilde{X}} & \xrightarrow{\sim} & \boldsymbol{\kappa}|_{\widetilde{X}} \\ \pm 1 & \mapsto & \pm 1 \\ \pm\sqrt{-1} & \mapsto & \pm\epsilon \end{array}$$

of group schemes in the étale topology on $\widetilde{X}$, descending to an isomorphism $\boldsymbol{\mu}_4 \xrightarrow{\sim} \boldsymbol{\kappa}$ over $X$. As before, this isomorphism depends on our particular choice of $\epsilon$.

**Theorem 2.11.** *Let $X$ be a noetherian scheme with the Krull-Schmidt property and with $\frac{1}{2} \in \mathscr{O}_X$. Let $(\mathscr{H}, h)$ be an $\mathscr{O}_X$-valued symmetric bilinear form on $X$ and $\boldsymbol{\kappa} = \boldsymbol{\kappa}(\mathscr{H}, h)$ is the kernel of the group scheme homomorphism $s : \mathbf{\Gamma}(\mathscr{H}, h) \to \mathbf{GO}(\mathscr{H}, h)$. Then the class of $\boldsymbol{\kappa}$ as defined in (2.12), is given by*

$$[\boldsymbol{\kappa}] = w_1(\mathscr{H}, h) + (-1) = w_1(\det \mathscr{H}, -\det h)$$

*in $H^1_{\text{ét}}(X, \boldsymbol{\mu}_2)$.*

As an example, the calculation in Example 2.10 reaffirms Proposition 2.11 for hyperbolic spaces with trivial lagrangians. Indeed, for any $m$ note that

$$[\boldsymbol{\kappa}] = \left\{ \begin{array}{rl} [\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2] = 1 & \text{if } m \text{ odd} \\ [\boldsymbol{\mu}_4] = (-1) & \text{if } m \text{ even} \end{array} \right\} = (-(-1)^m) = w_1(\mathscr{H}, h) + (-1)$$

in $H^1_{\text{ét}}(X, \boldsymbol{\mu}_2)$.

*Proof.* Since all terms in the formula are invariant under base change, we can use the orthogonal splitting principle, see §2.1.1, in particular we must assume $X$ has the Krull-Schmidt property. First, we will handle the case where $(\mathscr{H}, h)$ is isometric to a diagonal form $\perp_{k=1}^{n} (\mathscr{L}_k, l_k)$, with

$l_k : \mathscr{L}_k^{\otimes 2} \xrightarrow{\sim} \mathscr{O}_X$. The general case is a mixture of the calculation below that that performed in Example 2.10. Let $(\mathscr{L}, l) = \otimes_{k=1}^n (\mathscr{L}_k, l_k) = \det(\mathscr{H}, h)$ and define $p : \widetilde{X} \to X$ to be the étale double cover defined by $(\mathscr{L}, -l)$. We'll prove that $\boldsymbol{\kappa}$ trivializes over $\widetilde{X}$ by finding a section (over $\widetilde{X}$) of the exact sequence (2.11).

Let $\mathscr{U} = \{U_i \to X\}_{i \in I}$ be a Zariski open cover of $X$, temporarily denoting the indexing of the opens in superscripts, that trivializes the line bundles $\mathscr{L}_k$ via $\varphi_i^k : \mathscr{O}_{U_i} \xrightarrow{\sim} \mathscr{L}_k|_{U_i}$. Let $\Phi_i : \mathscr{O}_{U_i}^n \xrightarrow{\sim} \mathscr{H}|_{U_i}$ be the induced trivialization of $\mathscr{H}$. Then the induced forms, $(\mathscr{O}_{U_i}, l_k|_{U_i} \circ \varphi_i^k \otimes \varphi_i^k)$, of rank 1, correspond to sections $\varphi_i^{k\vee}\, \psi_{l_k|_{U_i}}\, \varphi_i^k = b_i^k \in \mathbb{G}_{\mathrm{m}}(U_i)$, and the induced form $(\mathscr{O}_{U_i}^n, h|_{U_i} \circ \Phi_i \otimes \Phi_i)$ is isometric to the diagonal form $\langle b_i^1, \ldots, b_i^n \rangle$ with orthogonal basis $e_i^1, \ldots, e_i^n$.

Define $d_i = b_i^1 \cdots b_i^n \in \mathbb{G}_{\mathrm{m}}(U_i)$, define a new (Zariski) cover $\widetilde{\mathscr{U}}$ where $\widetilde{U}_i = U_i \times_{\operatorname{Spec} \mathbb{Z}[\frac{1}{2}]} \operatorname{Spec} \mathbb{Z}[\frac{1}{2}, \sqrt{-d_i}]$, and finally let

$$\epsilon_i = \frac{1}{d_i} e_i^1 \cdots e_i^n \otimes \sqrt{-d_i} \in \mathbf{\Gamma}(\widetilde{U}_i).$$

Notice that (as in the proof of Proposition 2.8),

$$N(\epsilon_i) = -1 \in \mathbb{G}_{\mathrm{m}}(\widetilde{U}_i), \quad r(\epsilon_i) = -\mathrm{id} \in \mathbf{O}(\widetilde{U}_i),$$

so that $\epsilon_i \in \boldsymbol{\kappa}(U_i')$ defines a section of the exact sequence (2.11), for each $i \in I$. Noting that $p^{-1}U_i \cong \widetilde{U}_i$ forms a (Zariski) open cover of $\widetilde{X}$, we'll show that the sections $\epsilon_i$ glue to a section $\epsilon \in \boldsymbol{\kappa}(\widetilde{X})$.

For each $i, j \in I$, define $a_{ij}^k = {\varphi_{i\underline{j}}^k}^{-1}\, \varphi_{\underline{i}j}^k \in \mathbb{G}_{\mathrm{m}}(U_{ij})$, i.e. the Čech 1-cocycle representing $\mathscr{L}_k$. Then simple calculations show that

$$(a_{ij}^k)^2 = {b_{i\underline{j}}^k}^{-1}\, b_{\underline{i}j}^k, \qquad e_{i\underline{j}}^k = a_{ij}^k\, e_{\underline{i}j}^k, \qquad d_{i\underline{j}} = (a_{ij})^2\, d_{\underline{i}j},$$

where we set $a_{ij} = a_{ij}^1 \cdots a_{ij}^n \in \mathbb{G}_{\mathrm{m}}(U_{ij})$. Thus we have the equality

$$\begin{aligned} \epsilon_{i\underline{j}} &= \frac{1}{d_{i\underline{j}}} e_{i\underline{j}}^1 \cdots e_{i\underline{j}}^n \otimes \sqrt{-d_{i\underline{j}}} \\ &= \frac{1}{(a_{ij})^2\, d_{\underline{i}j}} a_{ij}^1\, e_{\underline{i}j}^1 \cdots a_{ij}^n\, e_{\underline{i}j}^n \otimes \sqrt{(a_{ij})^2\, d_{\underline{i}j}} \\ &= \frac{1}{d_{\underline{i}j}} e_{\underline{i}j}^1 \cdots e_{\underline{i}j}^n \otimes \sqrt{-d_{\underline{i}j}} = \epsilon_{\underline{i}j} \end{aligned}$$

of section in $\mathbf{\Gamma}(\widetilde{U}_{ij})$. Thus the sections $\epsilon_i \in \mathbf{\Gamma}(\widetilde{U}_i)$ glue to a section $\epsilon \in \mathbf{\Gamma}(\widetilde{X})$, which defines trivialization of exact sequence (2.11) over $\widetilde{X}$, i.e. $\boldsymbol{\kappa}(\widetilde{X})$ is isomorphic to $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, depending on $n$ modulo 4.

So far, we have only proved that the class of $\boldsymbol{\kappa}$ is in the subgroup (of order 2) generated by $w_1(\det \mathscr{H}, -\det h)$. We'll now prove that, at least over a cover of $X$ containing a square root of $-1$, $\boldsymbol{\kappa}$ does not trivialize if and only if $w_1(\det \mathscr{H}, \det h)$ is nontrivial. Let $X' \to X$ be the (possibly trivial) cover of $X$ taking a square root of $-1$. Then applying cohomology to diagram (2.10), a simple diagram chase shows that the image of $-1 \in \boldsymbol{\mu}_2(X')$ in $H^1_{\text{ét}}(X', \boldsymbol{\mu}_2)$ via the first coboundary map of the first column is equal to the image of $-\mathrm{id} \in \mathbf{O}(X')$ in $H^1_{\text{ét}}(X', \boldsymbol{\mu}_2)$ via the first coboundary map (the "spinor norm" $sp$) of the first row. Then $sp(-\mathrm{id})$ is nontrivial if and only if the first coboundary of the first column of (2.10) is nontrivial, which is equivalent to the nontriviality of $\boldsymbol{\kappa}$ over $X'$. Thus we are interesting in computing $sp(-\mathrm{id})$.

An easy generalization of Fröhlich [18, (I.4)] or Kahn [28, Lemma 2.1] is that for any $U \to X$, $sp(-\mathrm{id}) = w_1(\mathscr{L}_i|_U, l_i|_U)$ for the spinor norm $sp : \mathbf{O}(\mathscr{L}_i, l_i)(U) \to H^1_{\text{ét}}(U, \boldsymbol{\mu}_2)$ of a form $(\mathscr{L}_i, l_i)$ of rank 1. Combining this with a multiplicativity formula for the spinor norm of an orthogonal sum, see Fröhlich [18, (1.5)], if $(\mathscr{H}, h)$ is isometric to $\perp_{i=1}^n (\mathscr{L}_i, l_i)$, then $sp(-\mathrm{id}) = \sum_{i=1}^n w_1(\mathscr{L}_i|_{X'}, l_i|_{X'}) = w_1(\mathscr{H}|_{X'}, h|_{X'})$. Thus over $X'$ (i.e. if there's a square root of $-1$) the nontriviality of $\boldsymbol{\kappa}$ is equivalent to the triviality of $w_1(\mathscr{H}, h)$. The fact that we had to work over $X'$ leaves only the cases when either $w_1(\det \mathscr{H}, \det h)$ is $(-1)$ or trivial, which can be done by hand. We wonder, however, if the restriction to $X'$ was necessary in the first place. □

We ask if the formula in Theorem 2.11 is true without the assumption that $X$ satisfies the Krull-Schmidt property?

*Remark* 2.12. Using the other convention for the Clifford norm mentioned in Remark 2.9, the class of $\boldsymbol{\kappa}(\mathscr{H}, h)$ is then $w_1(\det \mathscr{H}, (-1)^{n+1} \det h) \in H^1_{\text{ét}}(X, \boldsymbol{\mu}_2)$.

### The four-fold cover

By the calculations in the proof of Theorem 2.11, if $(\mathscr{H}, h)$ has rank $n$ then via $\boldsymbol{\kappa} \hookrightarrow \boldsymbol{\Gamma}$, any section of $\boldsymbol{\kappa}$ not in the canonical subgroup $\boldsymbol{\mu}_2 \hookrightarrow \boldsymbol{\kappa}$ has degree the parity of $n$. Hence by the structure theory of the center of the Clifford algebra, see §1.32, the exact sequence of group schemes

$$1 \to \boldsymbol{\kappa} \to \boldsymbol{\Gamma} \to \mathbf{GO} \to 1, \tag{2.13}$$

is central only for $n$ odd. For $n$ even, its restricting to the even Clifford group,

$$1 \to \boldsymbol{\kappa} \to \mathbf{S\Gamma} \to \mathbf{GSO} \to 1, \tag{2.14}$$

is central. Depending on the parity of the rank, we will call the respective central sequence above the *Clifford sequence for the orthogonal similitude group*. For the case of even rank, we'll need to consider the restriction of the fundamental sequence to the even Clifford group,

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \boldsymbol{\mu}_2 \arrow[r] \arrow[d, "\iota"] & \mathbf{Spin} \arrow[r, "r"] \arrow[d] & \mathbf{SO} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \boldsymbol{\kappa} \arrow[r] \arrow[d] & \mathbf{S\Gamma} \arrow[r, "s"] \arrow[d, "N"] & \mathbf{GSO} \arrow[r] \arrow[d, "\mu"] & 1 \\
1 \arrow[r] & \boldsymbol{\mu}_2 \arrow[r] \arrow[d] & \mathbb{G}_{\mathrm{m}} \arrow[r, "2"] \arrow[d] & \mathbb{G}_{\mathrm{m}} \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd} \tag{2.15}$$

Now, we are ready to define the cohomological invariants we'll be primarily concerned with.

### 2.1.4 The similarity 2nd Hasse-Witt invariant

Recall, from Theorems 1.4 and 1.14, the description of the categories of $\mathbf{GO}$-torsors and $\mathbf{GSO}$-torsors over $X$.

**Theorem 2.13.** *Let $X$ be a scheme with $\frac{1}{2} \in \mathscr{O}_X$ and endowed with the étale topology. We fix an $\mathscr{O}_X$-valued symmetric bilinear space $(\mathscr{H}, h)$ of rank $n$ over $X$.*

*a) The category of $\mathbf{GO}$-torsors is equivalent to the category whose objects are all symmetric bilinear spaces of rank $n$ with values in a line bundle and whose morphisms are similarity transformations. In particular, $H^1_{\text{ét}}(X, \mathbf{GO})$ is in bijection with the set of similarity classes of line bundle-valued symmetric bilinear spaces of rank $n$ with distinguished point the similarity class of $(\mathscr{H}, h, \mathscr{O}_X)$.*

*b) Let $n$ be even. The category of $\mathbf{GSO}$-torsors is equivalent to the category whose objects are pairs $((\mathscr{E}, b, \mathscr{L}), \zeta)$, consisting of an $\mathscr{L}$-valued symmetric bilinear space of rank $n$ (for some line bundle $\mathscr{L}$ on $X$) together with an orientation isometry $\zeta : \operatorname{disc}(\mathscr{H}, b, \mathscr{O}_X) \to \operatorname{disc}(\mathscr{E}, b, \mathscr{L})$ of discriminant forms, and where morphisms between objects $((\mathscr{E}, b, \mathscr{L}), \zeta)$ and $((\mathscr{E}', b', \mathscr{L}'), \zeta')$ are similarity transformations $\varphi : (\mathscr{E}, b, \mathscr{L}) \to (\mathscr{E}', b', \mathscr{L}')$ such that $\zeta' = \operatorname{disc}(\varphi) \circ \zeta$. In particular $H^1_{\text{ét}}(X, \mathbf{GSO})$ is in bijection with the set of similarity classes of oriented line bundle-valued symmetric bilinear spaces of rank $n$ with distinguished point the similarity class of $(\mathscr{H}, h, \mathscr{O}_X)$ oriented by the identity map.*

**Definition 2.14.** Let $(\mathscr{H}, h)$ be an $\mathscr{O}_X$-valued symmetric bilinear space and $(\mathscr{E}, b, \mathscr{L})$ be an $\mathscr{L}$-valued symmetric bilinear space, both of rank $n$ on $X$. There is an important dichotomy in the definition of the $(\mathscr{H}, h)$-base pointed *similarity 2nd Hasse-Witt invariant*:

*a)* for $n$ odd, it's the coboundary map arising from the Clifford sequence for the orthogonal similitude group (2.13)

$$\begin{array}{rcl} H^1_{\text{ét}}(X, \mathbf{GO}) & \to & H^2_{\text{ét}}(X, \boldsymbol{\kappa}) \\ {[\mathscr{E}, b, \mathscr{L}]} & \mapsto & gw_2^{\mathscr{H}}(\mathscr{E}, b, \mathscr{L}), \end{array}$$

*b)* for $n$ even, it's the coboundary map arising from the (even) Clifford sequence for the orthogonal similitude group (2.14)

$$\begin{array}{rcl} H^1_{\text{ét}}(X, \mathbf{GSO}) & \to & H^2_{\text{ét}}(X, \boldsymbol{\kappa}) \\ {[\mathscr{E}, b, \mathscr{L}, \zeta]} & \mapsto & gw_2^{\mathscr{H}}(\mathscr{E}, b, \mathscr{L}, \zeta), \end{array}$$

where $\zeta : \operatorname{disc}(\mathscr{H}, h, \mathscr{O}_X) \xrightarrow{\sim} \operatorname{disc}(\mathscr{E}, b, \mathscr{L})$ is an orientation.

Note that for $n$ even, $gw_2^{\mathscr{H}}(\mathscr{E}, b, \mathscr{L}, \zeta)$ is a secondary characteristic class, since it's only defined when $gw_1^{\mathscr{H}}(\mathscr{E}, b, \mathscr{L})$ vanishes. Equivalently, it's only defined on the second level of the "fundamental filtration" of the Grothendieck-Witt group of line-bundle valued forms.

*Remark* 2.15. One should really be able to understand the similarity 2nd Hasse-Witt invariant, in the spirit of Jardine [27], as a universal characteristic class in the étale cohomology group $H^2_{\text{ét}}(B\mathbf{GSO}, \boldsymbol{\kappa})$ of the simplicial classifying scheme or classifying topos for the proper orthogonal similitude (in the even rank case) group scheme. There is thus the possibility of even higher similarity Hasse-Witt invariants. In the even rank case, the complexity of the structure of this cohomology is hinted at by the calculations (with $X = \operatorname{Spec}\mathbb{C}$) of Holla/Nitsure [25] and [26].

**Functoriality**

Let $f : Y \to X$ be a morphism of schemes in which 2 is invertible. Let $(\mathscr{E}, b, \mathscr{L})$ be an $\mathscr{L}$-valued symmetric bilinear space of rank $n$ on $X$, then $f^*(\mathscr{E}, b, \mathscr{L})$ is a canonical $f^*\mathscr{L}$-valued symmetric bilinear space of rank $n$ on $Y$. There's now a canonical isomorphism of group schemes $\boldsymbol{\kappa}(f^*(\mathscr{H}, h)) \xrightarrow{\sim} f^p\boldsymbol{\kappa}(\mathscr{H}, h)$ (where $f^p$ is the pullback in the category of étale sheaves, see Milne [36, II §2]). If $n$ is even and $\zeta : \operatorname{disc}(\mathscr{H}, h, \mathscr{O}_X) \xrightarrow{\sim} \operatorname{disc}(\mathscr{E}, b, \mathscr{L})$ is an orientation, then there's a canonical isomorphism of discriminant forms $f^* \operatorname{disc}(\mathscr{E}, b, \mathscr{L}) \xrightarrow{\sim} \operatorname{disc}(f^*(\mathscr{E}, b, \mathscr{L}))$, making $f^*\zeta : f^* \operatorname{disc}(\mathscr{H}, h, \mathscr{O}_X) \xrightarrow{\sim} f^* \operatorname{disc}(\mathscr{E}, b, \mathscr{L})$ into an orientation on $f^*(\mathscr{E}, b, \mathscr{L})$.

Finally, if $n$ is odd, then

$$gw_2^{f^*\mathscr{H}}(f^*(\mathscr{E}, b, \mathscr{L})) = f^* gw_2^{\mathscr{H}}(\mathscr{E}, b, \mathscr{L})$$

while if $n$ is even then

$$gw_2^{f^*\mathscr{H}}(f^*(\mathscr{E}, b, \mathscr{L}), f^*\zeta) = f^* gw_2^{\mathscr{H}}(\mathscr{E}, b, \mathscr{L}, \zeta)$$

in $H^2_{\text{ét}}(Y, \boldsymbol{\kappa}(f^*(\mathscr{H}, h))) \xrightarrow{\sim} H^2_{\text{ét}}(Y, f^p\boldsymbol{\kappa}(\mathscr{H}, h))$.

**Interpolation property**

The similarity 2nd Hasse-Witt invariant "interpolates" between the classical 2nd Hasse-Witt invariant and the 1st Chern class modulo 2.

**Proposition 2.16.** *Let $(\mathscr{H}, h)$ be an $\mathscr{O}_X$-valued symmetric bilinear space and $(\mathscr{E}, b, \mathscr{L})$ be an $\mathscr{L}$-valued symmetric bilinear space, both of rank $n$ on $X$.*

*a) If $\mathscr{L} = \mathscr{O}_X$ is trivial then the similarity 2nd Hasse-Witt invariant $gw_2^{\mathscr{H}}(\mathscr{E}, b, \mathscr{O}_X)$ (resp. $gw_2^{\mathscr{H}}(\mathscr{E}, b, \mathscr{O}_X, \zeta)$ for any orientation $\zeta$, in the case of even rank), coincides with the image of the 2nd Hasse-Witt invariant $w_2^{\mathscr{H}}(\mathscr{E}, b)$ (with no dependence on the orientation), under the canonical map $\iota_* : H^1_{\text{ét}}(X, \boldsymbol{\mu}_2) \to H^1_{\text{ét}}(X, \boldsymbol{\kappa})$.*

*b) Under the induced map $H^1_{\text{ét}}(X, \boldsymbol{\kappa}) \to H^1_{\text{ét}}(X, \boldsymbol{\mu}_2)$, the similarity 2nd Hasse-Witt invariant $gw_2^{\mathscr{H}}(\mathscr{E}, b, \mathscr{L})$ (resp. $gw_2^{\mathscr{H}}(\mathscr{E}, b, \mathscr{O}_X, \zeta)$ for any orientation $\zeta$, in the case of even rank), maps to the first Chern class $c_1(\mathscr{L}, \boldsymbol{\mu}_2)$ modulo 2.*

*Proof.* One needs only to consider the implications on étale cohomology of the fundamental diagrams (2.10) and (2.15). □

**Comparing base forms**

In analogy with Proposition 2.1, we compare similarity 2nd Hasse-Witt invariants with different base forms. This is more subtle than in the classical case due to the dependence of $\boldsymbol{\kappa}$ on the base form. Because of our method of proof (we use the orthogonal splitting principle), we must restrict to the case of schemes $X$ that satisfy the Krull-Schmidt property. We do not believe this restriction to be necessary. We'll first need an additional construction.

**Lemma 2.17.** *Let $X$ be a scheme with the Krull-Schmidt property and with $\frac{1}{2} \in \mathscr{O}_X$. Let $\psi : \mathrm{disc}(\mathscr{H}, h, \mathscr{O}_X) \xrightarrow{\sim} \mathrm{disc}(\mathscr{H}', h', \mathscr{O}_X)$ be an isometry of discriminant forms of $\mathscr{O}_X$-valued symmetric bilinear spaces of rank $n$ on $X$. Letting $\boldsymbol{\kappa} = \boldsymbol{\kappa}(\mathscr{H}, h)$ and $\boldsymbol{\kappa}' = \boldsymbol{\kappa}(\mathscr{H}', h')$, then for all $i \geq 0$ there's an induced isomorphism of étale cohomology groups*

$$\psi_* : H^i_{\text{ét}}(X, \boldsymbol{\kappa}) \to H^i_{\text{ét}}(X, \boldsymbol{\kappa}').$$

*Proof.* First, we will handle the case where we can find a splitting morphism $f : Y \to X$ such that $f^*(\mathscr{H}, h)$ and $f^*(\mathscr{H}', h')$ are both isometric to a diagonal forms $\perp_{k=1}^n (\mathscr{L}_k, l_k)$ and $\perp_{k=1}^n (\mathscr{L}'_k, l'_k)$, respectively. The case where the forms contain hyperbolic orthogonal summands in their splittings is similar. As in the proof of Theorem 2.11, let $\mathscr{U} = \{U_i \to Y\}_{i \in I}$ be a Zariski cover of $Y$ trivializing $f^*\mathscr{H}$ and $f^*\mathscr{H}'$ via $\Phi_i : \mathscr{O}^n_{U_i} \xrightarrow{\sim} f^*\mathscr{H}|_{U_i}$ and $\Phi'_i : \mathscr{O}^n_{U_i} \xrightarrow{\sim} f^*\mathscr{H}'|_{U_i}$, so that the induced forms $(\mathscr{O}^n_{U_i}, h|_{U_i} \circ \Phi_i \otimes \Phi_i)$ and $(\mathscr{O}^n_{U_i}, h'|_{U_i} \circ \Phi'_i \otimes \Phi'_i)$ are isometric to diagonal forms $\langle b^1_i, \ldots, b^n_i \rangle$ and $\langle b'^1_i, \ldots, b'^n_i \rangle$, respectively, with global orthogonal bases $e^1_i, \ldots, e^n_i$ and $e'^1_i, \ldots, e'^n_i$. Also let $\psi_i = \det \Phi'^{-1}_i \circ \psi|_{U_i} \circ \det \Phi_i$ be the isometry of discriminant forms induced from $\psi$ and define $u_i \in \mathbb{G}_\mathrm{m}(U_i)$ by the equation

$$\psi_i(e^1_i \wedge \cdots \wedge e^n_i) = u_i \, e'^1_i \wedge \cdots \wedge e'^n_i.$$

Then by a calculation, we have $d_i = u_i^2 d'_i$ where $d_i = b^1_i \cdots b^n_i$ and $d'_i = b'^1_i \cdots b'^n_i$. Let $\epsilon_i \in f^p\boldsymbol{\kappa}(\widetilde{U}_i)$ and $\epsilon'_i \in f^p\boldsymbol{\kappa}'(\widetilde{U}'_i)$ be the sections over $\widetilde{U}_i = U_i \times_{\operatorname{Spec}\mathbb{Z}[\frac{1}{2}]} \operatorname{Spec}\mathbb{Z}[\frac{1}{2}, \sqrt{-d_i}]$ and $\widetilde{U}'_i = U_i \times_{\operatorname{Spec}\mathbb{Z}[\frac{1}{2}]} \operatorname{Spec}\mathbb{Z}[\frac{1}{2}, \sqrt{-d'_i}]$, defined by

$$\epsilon_i = \frac{1}{d_i} e^1_i \cdots e^n_i \otimes \sqrt{-d_i} \in f^p\boldsymbol{\kappa}(\widetilde{U}_i), \quad \epsilon'_i = \frac{1}{d'_i} e'^1_i \cdots e'^n_i \otimes \sqrt{-d'_i} \in f^p\boldsymbol{\kappa}'(\widetilde{U}'_i).$$

Finally note that we have the equality $u_i \, \Psi_i(\epsilon_i) = \epsilon'_i$ of sections in $f^p\boldsymbol{\kappa}'(\widetilde{U}_i \times_{U_i} \widetilde{U}'_i)$, where $\Psi_i$ is the Clifford algebra homomorphism induced from $e^k_i \mapsto e'^k_i$. The family of maps $(u_i \, \Psi_i : f^p\boldsymbol{\kappa}|_{U_i} \to f^p\boldsymbol{\kappa}'|_{U_i})_{i \in I}$ are then seen to glue to a group scheme isomorphism $\Psi : f^p\boldsymbol{\kappa} \to f^p\boldsymbol{\kappa}'$ over $Y$.

Now given $\alpha \in H^i_{\text{ét}}(X, \boldsymbol{\kappa})$, the element $H^i_{\text{ét}}(\Psi)(f^*\alpha) \in H^i_{\text{ét}}(Y, f^p\boldsymbol{\kappa}')$ is then seen to be in the image of $f^* : H^i_{\text{ét}}(X, \boldsymbol{\kappa}') \to H^i_{\text{ét}}(Y, \boldsymbol{\kappa}')$, hence (by the injectivity of $f^*$) defines an element $\psi_*\alpha \in H^i_{\text{ét}}(X, \boldsymbol{\kappa}')$. This defines a homomorphism of abelian groups $\psi_* : H^i_{\text{ét}}(X, \boldsymbol{\kappa}) \to H^i_{\text{ét}}(X, \boldsymbol{\kappa}')$. By construction, $(\psi^{-1})_* = (\psi_*)^{-1}$ so the map is also an isomorphism. $\square$

**Theorem 2.18.** *Let $X$ be a scheme with the Krull-Schmidt property and with $\frac{1}{2} \in \mathscr{O}_X$. Let $(\mathscr{H}, h)$, $(\mathscr{H}', h')$ be $\mathscr{O}_X$-valued symmetric bilinear spaces and $(\mathscr{E}, b, \mathscr{L})$ be an $\mathscr{L}$-valued symmetric bilinear space, all of rank $n$ on $X$. Then*

$$gw_1^{\mathscr{H}'}(\mathscr{E}, b, \mathscr{L}) = w_1^{\mathscr{H}'}(\mathscr{H}, h) + gw_1^{\mathscr{H}}(\mathscr{E}, b, \mathscr{L})$$

*in* $H^1_{\text{ét}}(X, \boldsymbol{\mu}_2)$. *Furthermore, if* $gw_1^{\mathscr{H}'}(\mathscr{E}, b, \mathscr{L})$ *and* $gw_1^{\mathscr{H}}(\mathscr{E}, b, \mathscr{L})$ *are trivial, and we choose orientations* $\zeta : \mathrm{disc}(\mathscr{H}, h, \mathscr{O}_X) \xrightarrow{\sim} \mathrm{disc}(\mathscr{E}, b, \mathscr{L})$ *and* $\zeta' : \mathrm{disc}(\mathscr{H}', h', \mathscr{O}_X) \xrightarrow{\sim} \mathrm{disc}(\mathscr{E}, b, \mathscr{L})$, *then*

$$gw_2^{\mathscr{H}'}(\mathscr{E}, b, \mathscr{L}, \zeta') = \iota_* w_2^{\mathscr{H}'}(\mathscr{H}, h) + (\zeta'^{-1} \circ \zeta)_* gw_2^{\mathscr{H}}(\mathscr{E}, b, \mathscr{L}, \zeta)$$

*in* $H^2_{\text{ét}}(X, \boldsymbol{\kappa}')$, *where* $(\zeta'^{-1} \circ \zeta)_*$ *is the homomorphism on cohomology defined above.*

*Proof.* The argument is by explicit cocycle calculation, using a variant of Cassou-Noguès/Erez/Taylor [12, Theorem 0.2]. $\square$

Though there is no direct comparison theorem for invariants of forms of odd rank, there is really no need for one, as we have an exact formula for the invariants in this case, see Theorem 2.19.

## 2.2 Explicit calculations

First, we will dispense with the case of $\mathscr{L}$-valued symmetric bilinear spaces of odd rank, for which is the similarity 2nd Hasse-Witt invariant is expressed in terms of classical invariants. It's no surprise that this case is easy to deal with, considering the simple structure of the group scheme of odd rank orthogonal similitudes, see 1 §1.3.1. Next, we will treat the case of metabolic forms, whose similarity 2nd Hasse-Witt invariants are calculated using an explicit cocycle calculation in the Clifford group.

### 2.2.1 The odd rank case

The calculation of the similarity 2nd Hasse-Witt invariant $gw_2^{\mathscr{H}}$ for odd rank forms reduces to the calculation of Chern classes modulo 2, the discriminant of the base form, and the classical 2nd Hasse-Witt invariants $w_2^{\mathscr{H}}$ of normalized forms.

Let $(\mathscr{H}, h)$ be an $\mathscr{O}_X$-valued symmetric bilinear form of odd rank $n = 2m + 1$ on $X$. We'll briefly recall the results from §1.3.1 on the structure of the odd rank orthogonal similitude group.

For every $\mathscr{L}$-valued symmetric bilinear form $(\mathscr{E}, b, \mathscr{L})$ of odd rank $n$ on $X$, we know that $\mathscr{L}$ is a square in the Picard group (see Theorem 1.15). Moreover, there's a canonical choice of $\mathscr{L}$-valued line called the *absolute value form* $|\mathscr{E}, b, \mathscr{L}|$. The *normalized form* $u(\mathscr{E}, b, \mathscr{L})$ is an $\mathscr{O}_X$-valued space of rank $n$ with trivial discriminant form (see Lemma 1.20), and such that there's a canonical isometry of $\mathscr{L}$-valued forms $|\mathscr{E}, b, \mathscr{L}| \otimes u(\mathscr{E}, b, \mathscr{L}) \xrightarrow{\sim} (\mathscr{E}, b, \mathscr{L})$. We recall that the associated group scheme homomorphisms,

$$\mathbf{GO}(\mathscr{H}, h) \xrightarrow{|\cdot|} \mathbf{GL}(|\mathscr{H}, h, \mathscr{O}_X|) = \mathbb{G}_{\mathrm{m}}, \qquad \mathbf{GO}(\mathscr{H}, h) \xrightarrow{u} \mathbf{SO}(u(\mathscr{H}, h, \mathscr{O}_X)),$$

induce the maps $[\mathscr{E}, b, \mathscr{L}] \mapsto |\mathscr{E}, b, \mathscr{L}| \otimes |\mathscr{H}, h, \mathscr{O}_X|^\vee$ and $[\mathscr{E}, b, \mathscr{L}] \mapsto u(\mathscr{E}, b, \mathscr{L})$, respectively, on isomorphism classes of torsors. We can now state the main result on odd rank forms.

**Theorem 2.19.** *Let* $(\mathscr{E}, b, \mathscr{L})$ *be a symmetric bilinear space of odd rank* $n = 2m+1$ *on* $X$. *Then*

$$gw_2^{\mathscr{H}}(\mathscr{E}, b, \mathscr{L}) = \iota_*\big(c_1(\mathscr{E}, \boldsymbol{\mu}_2) + (m+1)\,(-1, -1) + w_2^{\mathscr{H}}(u(\mathscr{E}, b, \mathscr{L}))\big)$$

*in* $H^2_{\text{ét}}(X, \boldsymbol{\kappa})$ *where* $(-1, -1) = (-1) \cdot (-1)$ *is the standard cup product symbol.*

In the proof, we will need the following lemmas about the classical Hasse-Witt invariants. The first concerns the Hasse-Witt invariants of a form tensor a line.

**Lemma 2.20.** *Let $(\mathscr{H}, h)$ be an $\mathscr{O}_X$-valued symmetric bilinear form of rank $n$ on $X$ and $(\mathscr{L}, l)$ an $\mathscr{O}_X$-valued line. Then we have*

$$w_2(\mathscr{L}\otimes\mathscr{H}, l\otimes h) = \frac{n(n-1)}{2} w_1(\mathscr{L}, l)\cdot w_1(\mathscr{L}, l) + (n-1)w_1(\mathscr{L}, l)\cdot w_1(\mathscr{H}, h) + w_2(\mathscr{H}, h)$$

*in $H^2_{\text{ét}}(X, \boldsymbol{\mu}_2)$.*

*Remark* 2.21. In general, in terms of the formal Hasse-Witt polynomial

$$w_t(\mathscr{H}, h) = \sum_{i=0}^{n} w_i(\mathscr{H}, h)\, t^{n-i}$$

we have that in terms of the total Hasse-Witt invariant,

$$w(\mathscr{L}\otimes\mathscr{H}, l\otimes h) = w_{1+w_1(\mathscr{L},l)}(\mathscr{H}, h),$$

in $H^*(X, \boldsymbol{\mu}_2)$, of which the above formula is just the degree two part.

The second compares the 1st Hasse-Witt invariant and 1st Chern class modulo 2, see Esnault/Kahn/Viehweg [17, Lemma 5.3] for a proof.

**Lemma 2.22.** *Let $(\mathscr{L}, l)$ be an $\mathscr{O}_X$-valued line. Then*

$$c_1(\mathscr{L}, \boldsymbol{\mu}_2) = w_1(\mathscr{L}, l)\cdot w_1(\mathscr{L}, l) + (-1)\cdot w_1(\mathscr{L}, l)$$

*in $H^2_{\text{ét}}(X, \boldsymbol{\mu}_2)$.*

*Proof of Theorem 2.19.* Consider the implications on étale cohomology of the following commutative diagram of group schemes with exact rows and columns,

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
1 \arrow[r] & \boldsymbol{\mu}_2 \arrow[r] \arrow[d] & \mathbf{S\Gamma} \arrow[r, "N\times r"] \arrow[d] & \mathbb{G}_{\mathrm{m}} \times \mathbf{SO} \arrow[r] \arrow[d, shift right, "m"'] & 1 \\
1 \arrow[r] & \boldsymbol{\kappa} \arrow[r] \arrow[d] & \mathbf{\Gamma} \arrow[r, "s"] \arrow[d] & \mathbf{GO} \arrow[r] \arrow[u, shift right, "|\cdot|\times u"'] & 1 \\
 & \boldsymbol{\mu}_2 \arrow[r, equal] \arrow[d] & \boldsymbol{\mu}_2 \arrow[d] & & \\
 & 1 & 1 & &
\end{tikzcd}$$

in the étale topology on $X$. By a "special" version (i.e. restricting to special and even subgroups) of Proposition 2.5 and the above diagram, we have

$$gw_2^{\mathscr{H}}(\mathscr{E}, b, \mathscr{L}) = \iota_*\Big(c_1(|\mathscr{E}, b, \mathscr{L}|\otimes|\mathscr{H}, h, \mathscr{O}_X|^\vee, \boldsymbol{\mu}_2) + w_2^{u(\mathscr{H})}(u(\mathscr{E}, b, \mathscr{L}))\Big),$$

and the rest of the argument is unwinding this. First, note that using §1.3.1, we have

$$\begin{aligned} c_1(|\mathscr{E}| \otimes |\mathscr{H}|^\vee, \boldsymbol{\mu}_2) &= c_1(\mathscr{L}^{\vee\otimes m} \otimes \det \mathscr{E} \otimes \det \mathscr{H}^\vee, \boldsymbol{\mu}_2) \\ &= m\, c_1(\mathscr{L}, \boldsymbol{\mu}_2) + c_1(\mathscr{E}, \boldsymbol{\mu}_2) + c_1(\mathscr{H}, \boldsymbol{\mu}_2) \\ &= c_1(\mathscr{E}, \boldsymbol{\mu}_2) + c_1(\mathscr{H}, \boldsymbol{\mu}_2), \end{aligned}$$

since $\mathscr{L}$ is a square in the Picard group, see Theorem 1.15. Now we deal the Hasse-Witt invariant term.

Let $(\mathscr{T}, t)$ be the standard sum-of-squares form of rank $n$ and recall that we've defined $w_i = w_i^{\mathscr{T}}$. For any $\mathscr{O}_X$-valued symmetric bilinear forms $(\mathscr{B}, b)$ and $(\mathscr{B}', b')$ of rank $n$, repeated use of the base form comparison formulas of Proposition 2.1 yields the general formulas

$$w_1^{\mathscr{B}'}(\mathscr{B}) = w_1(\mathscr{B}') + w_1(\mathscr{B})$$

and

$$\begin{aligned} w_2^{\mathscr{B}'}(\mathscr{B}) &= w_2^{\mathscr{B}'}(\mathscr{T}) + w_1^{\mathscr{B}'}(\mathscr{T}) \cdot w_1(\mathscr{B}) + w_2(\mathscr{B}) \\ &= w_2(\mathscr{B}') + w_1(\mathscr{B}') \cdot w_1(\mathscr{B}') + w_1(\mathscr{B}') \cdot w_1(\mathscr{B}) + w_2(\mathscr{B}), \end{aligned}$$

temporarily condensing our notation.

Now, using the base form comparison formula once, we have

$$w_2^{u(\mathscr{H})}(u(\mathscr{E}, b, \mathscr{L})) = w_2^{u(\mathscr{H})}(\mathscr{H}, h) + w_1^{u(\mathscr{H})}(\mathscr{H}, h) \cdot w_1^{\mathscr{H}}(u(\mathscr{E}, b, \mathscr{L})) + w_2^{\mathscr{H}}(u(\mathscr{E}, b, \mathscr{L})).$$

Applying the above formulas to the first two terms of the right-hand-side, noting that $w_1(u(\mathscr{H}, h))$ is trivial, yields

$$w_2^{u(\mathscr{H})}(\mathscr{H}, h) = w_2(u(\mathscr{H}, h)) + w_2(\mathscr{H}, h)$$

and

$$w_1^{u(\mathscr{H})}(\mathscr{H}, h) \cdot w_1^{\mathscr{H}}(u(\mathscr{E}, b, \mathscr{L})) = w_1(\mathscr{H}, h) \cdot w_1(\mathscr{H}, h).$$

By Lemma 2.20 and the fact that $n$ is odd, we have

$$w_2(u(\mathscr{H}, h)) = w_2(\det \mathscr{H}^\vee \otimes \mathscr{H}, \det h^\vee \otimes h) = m\, w_1(\mathscr{H}, h) \cdot w_1(\mathscr{H}, h) + w_2(\mathscr{H}, h),$$

and thus finally,

$$w_2^{u(\mathscr{H})}(u(\mathscr{E}, b, \mathscr{L})) = (m+1)\, w_1(\mathscr{H}, h) \cdot w_1(\mathscr{H}, h) + w_2^{\mathscr{H}}(u(\mathscr{E}, b, \mathscr{L})).$$

Putting everything together yields

$$gw_2^{\mathscr{H}}(\mathscr{E}, b, \mathscr{L}) = \iota_*\big(c_1(\mathscr{E}, \boldsymbol{\mu}_2) + c_1(\mathscr{H}, \boldsymbol{\mu}_2) + (m+1)\, w_1(\mathscr{H}, h) \cdot w_1(\mathscr{H}, h) + w_2^{\mathscr{H}}(u(\mathscr{E}, b, \mathscr{L}))\big).$$

Using Lemma 2.22, we have

$$\begin{aligned} \iota_* c_1(\mathscr{H}, \boldsymbol{\mu}_2) &= \iota_*\big((-1) \cdot w_1(\mathscr{H}, h) + w_1(\mathscr{H}, h) \cdot w_1(\mathscr{H}, h)\big) \\ &= \iota_*(-1) \cdot \iota_* w_1(\mathscr{H}, h) + \iota_* w_1(\mathscr{H}, h) \cdot \iota_* w_1(\mathscr{H}, h) \\ &= \iota_*\big((-1) \cdot (-1) + (-1) \cdot (-1)\big) \end{aligned}$$

which is trivial, since $\iota_* w_1(\mathscr{H}, h) = \iota_*(-1) \in H^1_{\text{ét}}(X, \boldsymbol{\kappa}(\mathscr{H}, h))$ by Theorem 2.11. Finally, we obtain the stated formula. □

### 2.2.2 Invariants of metabolic spaces

Recall from §1.4 the notion of $\mathscr{L}$-valued metabolic spaces on $X$. For the following, fix $(\mathscr{H}, h) = H_{\mathscr{O}_X}(\mathscr{O}_X^m)$ as the $n = 2m$ dimensional $\mathscr{O}_X$-valued hyperbolic space with trivial lagrangian and denote $\mathbf{GSO}_{m,m} = \mathbf{GSO}(\mathscr{H}, h)$. As a consequence of the following lemma, every oriented metabolic space is a $\mathbf{GSO}_{m,m}$-torsor.

**Lemma 2.23.** *Let $(\mathscr{E}, b, \mathscr{L})$ be a metabolic space of rank $n = 2m$ on $X$. Then any choice of lagrangian $\mathscr{V} \hookrightarrow \mathscr{E}$ induces an orientation $\zeta_{\mathscr{V}} : \langle(-1)^m\rangle \xrightarrow{\sim} \mathrm{disc}(\mathscr{E}, b, \mathscr{L})$.*

*Proof.* The proof is a straightforward adaptation of the classical case of $\mathscr{O}_X$-valued metabolic spaces, see Knebusch [30, IV Proposition 3.2]. Let $0 \to \mathscr{V} \xrightarrow{j} \mathscr{E} \xrightarrow{p} \mathscr{W} \to 0$ be the exact sequence of vector bundles associated to the lagrangian $\mathscr{V}$. First, we'll compute the determinant form of $(\mathscr{E}, b, \mathscr{L})$. There's a canonical isomorphism $\alpha : \det \mathscr{V} \otimes \det \mathscr{W} \to \det \mathscr{E}$ defined by

$$v_1 \wedge \cdots \wedge v_m \otimes w_1 \wedge \cdots \wedge w_m \mapsto v_1 \wedge \cdots \wedge v_m \wedge \overline{w}_1 \wedge \cdots \wedge \overline{w}_m$$

on sections over $U \to X$, where $U$ is fine enough so that lifts $\overline{w}_i \in \mathscr{E}(U)$ of $w_i \in \mathscr{W}(U)$ via $p$ exist. Since $\mathscr{V}$ is a lagrangian, the choice of lifts does not matter, and the map is well defined. There's also a canonical perfect pairing,

$$\begin{array}{ccc} \mathscr{V} \otimes \mathscr{W} & \to & \mathscr{L} \\ v \otimes w & \mapsto & b(v, \overline{w}) \end{array}$$

defined on section whenever $U \to X$ is fine enough so that $\overline{w} \in \mathscr{E}(U)$ is a lift of $w \in \mathscr{W}(U)$ via $p$. The pairing has an adjoint $\mathscr{O}_X$-module isomorphism $\psi : \mathscr{W} \to \mathscr{H}om(\mathscr{V}, \mathscr{L})$, and an induced isomorphism $\det \psi : \det \mathscr{W} \to \det \mathscr{H}om(\mathscr{V}, \mathscr{L})$ of determinants. Recall, from §1.2.1, that there's a canonical isomorphism $\mathrm{can} : \det \mathscr{H}om(\mathscr{V}, \mathscr{L}) \to \mathscr{H}om(\det \mathscr{V}, \mathscr{L}^{\otimes m})$. Then define an $\mathscr{O}_X$-module isomorphism $\zeta : \mathscr{L}^{\otimes m} \to \det \mathscr{E}$ via the commutativity of the following diagram,

$$\begin{tikzcd} \det \mathscr{V} \otimes \mathscr{H}om(\det \mathscr{V}, \mathscr{L}^{\otimes m}) \arrow[r, "\mathrm{ev}"] & \mathscr{L}^{\otimes m} \arrow[d, "\zeta"] \\ \det \mathscr{V} \otimes \det \mathscr{W} \arrow[u, "\mathrm{id} \otimes (\mathrm{can} \circ \det \psi)"] \arrow[r, "\alpha"] & \det \mathscr{E} \end{tikzcd}$$

of vector bundles on $X$. We now claim that $\zeta : (\mathscr{L}^{\otimes m}, \langle(-1)^m\rangle, \mathscr{L}^{\otimes n}) \to \det(\mathscr{E}, b, \mathscr{L})$ is an isometry of $\mathscr{L}^{\otimes n}$-valued lines. This may be verified locally just as in the classical case. Moving from determinants to discriminants, i.e. tensoring with $(\mathscr{L}^{\vee \otimes m}, \langle 1\rangle, \mathscr{L}^{\vee \otimes n})$, yields the desired orientation $\zeta_{\mathscr{V}} : \langle(-1)^m\rangle \xrightarrow{\sim} \mathrm{disc}(\mathscr{E}, b, \mathscr{L})$. □

*Remark* 2.24. As a consequence of the lemma, if $(\mathscr{H}, h)$ is metabolic (in particular hyperbolic) then $gw_1^{\mathscr{H}}(\mathscr{E}, b, \mathscr{L})$ is trivial as they have isometric discriminants.

Any hyperbolic space $H_{\mathscr{L}}(\mathscr{V})$ comes equipped with the orientation $\zeta_{\mathscr{V}}$. Note that under the switch isometry, the hyperbolic space $H_{\mathscr{L}}(\mathscr{H}om(\mathscr{V}, \mathscr{L}))$ has the opposite orientation. This means that while $H_{\mathscr{L}}(\mathscr{V})$ and $H_{\mathscr{L}}(\mathscr{H}om(\mathscr{V}, \mathscr{L}))$ are isomorphic as $\mathbf{GO}_{m,m}$-torsors, they represent (with their natural orientations) different isomorphism classes of $\mathbf{GSO}_{m,m}$-torsors.

**Group schemes isomorphic to $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$**

Denote $\boldsymbol{\kappa} = \boldsymbol{\kappa}_{m,m} = \boldsymbol{\kappa}(\mathscr{H}, h)$ and then recall that by Example 2.10,

$$\boldsymbol{\kappa}_{m,m} \cong \begin{cases} \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 & \text{if } m \text{ odd} \\ \boldsymbol{\mu}_4 & \text{if } m \text{ even} \end{cases}$$

In $m$ is odd, then by our running hypothesis that $X$ is connected, $\boldsymbol{\kappa} = \{\pm 1, \pm\epsilon\}$ is generated by global sections. Thus $\boldsymbol{\kappa}$ has a canonically determined subgroup scheme $\boldsymbol{\kappa}^0 = \{\pm 1\}$ and subgroup schemes $\boldsymbol{\kappa}^+ = \{1, \epsilon\}$ and $\boldsymbol{\kappa}^- = \{1, -\epsilon\}$ that are determined up to a choice of global section $\epsilon \in \boldsymbol{\kappa}(X)$. For the standard hyperbolic basis $e_1, \ldots, e_m, f_1, \ldots, f_m$, we will fix the choice of global section $\epsilon = \prod_{l=1}^{m}(1 - e_l f_l)$, to be the one from Example 2.10.

There's a split exact sequence,

$$1 \to \boldsymbol{\kappa} \xrightarrow{p^- \times p^0 \times p^+} \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 \xrightarrow{m} \boldsymbol{\mu}_2 \to 1,$$

well-defined up to ordering the three subgroups, where for $\bullet \in \{-, 0, +\}$, $p^\bullet$ is the quotient map,

$$1 \to \boldsymbol{\kappa}^\bullet \to \boldsymbol{\kappa} \xrightarrow{p^\bullet} \boldsymbol{\mu}_2 \to 1,$$

and $m$ is the total multiplication homomorphism. In particular, there are exact sequences,

$$0 \to H^i_{\text{ét}}(X, \boldsymbol{\kappa}) \to H^i_{\text{ét}}(X, \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2) \to H^i_{\text{ét}}(X, \boldsymbol{\mu}_2) \to 0.$$

for each $i \geq 0$. If $(\mathscr{E}, b, \mathscr{L})$ is an $(\mathscr{H}, h)$-oriented $\mathscr{L}$-valued symmetric bilinear space of rank $n \equiv 2 \bmod 4$ on $X$, then the formula

$$p^0_* gw_2^{\mathscr{H}}(\mathscr{E}, b, \mathscr{L}, \zeta) = c_1(\mathscr{L}, \boldsymbol{\mu}_2) \in H^2_{\text{ét}}(X, \boldsymbol{\mu}_2)$$

is a restatement of Proposition 2.16b. Thus by the above exact sequence of cohomology groups (for $i = 2$),

$$p^-_* gw_2^{\mathscr{H}}(\mathscr{E}, b, \mathscr{L}, \zeta) + p^+_* gw_2^{\mathscr{H}}(\mathscr{E}, b, \mathscr{L}, \zeta) = c_1(\mathscr{L}, \boldsymbol{\mu}_2) \in H^2_{\text{ét}}(X, \boldsymbol{\mu}_2).$$

In particular, the elements $p^+_* gw_2^{\mathscr{H}}(\mathscr{E}, b, \mathscr{L}, \zeta)$ and $c_1(\mathscr{L}, \boldsymbol{\mu}_2)$ in $H^2_{\text{ét}}(X, \boldsymbol{\mu}_2)$ completely determine the invariant $gw_2^{\mathscr{H}}(\mathscr{E}, b, \mathscr{L}, \zeta) \in H^2_{\text{ét}}(X, \boldsymbol{\kappa})$.

As for $m$ even, $\boldsymbol{\kappa} = \boldsymbol{\kappa}_{m,m} \cong \boldsymbol{\mu}_4$ by Example 2.10, and there is no method analogous to the above for identifying elements of $H^2_{\text{ét}}(X, \boldsymbol{\kappa})$. Instead, we'll introduce the following notation. For a line bundle $\mathscr{L}$ on $X$ and for $m$ even, define

$$c_1(\mathscr{L}, \boldsymbol{\kappa}) = c_1(\mathscr{L}, \boldsymbol{\kappa}_{m,m}) = gw_2^{\mathscr{H}}(H_{\mathscr{L}}(\mathscr{O}_X^m))$$

in $H^2_{\text{ét}}(X, \boldsymbol{\kappa})$, where as always $(\mathscr{H}, h) = H_{\mathscr{O}_X}(\mathscr{O}_X^m)$. The notation is justified since for any group scheme isomorphism $\boldsymbol{\mu}_4 \xrightarrow{\sim} \boldsymbol{\kappa}$, we have that $\pm c_1(\mathscr{L}, \boldsymbol{\mu}_4) \mapsto c_1(\mathscr{L}, \boldsymbol{\kappa})$ under the induced isomorphism $H^2_{\text{ét}}(X, \boldsymbol{\mu}_4) \xrightarrow{\sim} H^2_{\text{ét}}(X, \boldsymbol{\kappa})$.

Recall that when we speak of the similarity 2nd Hasse-Witt invariant of a hyperbolic form on a particular lagrangian $\mathscr{V}$, we are implicitly considering the form with the canonical orientation $\zeta_{\mathscr{V}}$ of Lemma 2.23.

**Theorem 2.25.** *Let $(\mathscr{H}, h) = H_{\mathscr{O}_X}(\mathscr{O}_X^m)$ and let $\mathscr{V}$ be a vector bundle of rank $m$ on $X$.*

*a) If $m$ is odd, then (using the notation of the above discussion)*

$$p_*^+ gw_2^{\mathscr{H}}(H_{\mathscr{L}}(\mathscr{V})) = c_1(\mathscr{V}, \boldsymbol{\mu}_2) + \frac{m+1}{2} c_1(\mathscr{L}, \boldsymbol{\mu}_2)$$

*in $H^2_{\text{ét}}(X, \boldsymbol{\mu}_2)$.*

*b) If $m$ is even, then*

$$gw_2^{\mathscr{H}}(H_{\mathscr{L}}(\mathscr{V})) = \iota_* \Big( c_1(\mathscr{V}, \boldsymbol{\mu}_2) + \frac{m}{2} c_1(\mathscr{L}, \boldsymbol{\mu}_2) \Big) + c_1(\mathscr{L}, \boldsymbol{\kappa}),$$

*in $H^2_{\text{ét}}(X, \boldsymbol{\kappa})$.*

*Proof.* We proceed by a Čech cohomology and Clifford algebra calculation, the spirit of which must be classical. But as no suitable reference exists in the literature, we include all the details here.

First, to fix the idea, we cover the case $m = 1$. Let $\mathscr{U} = \{U_i\}_{i \in I}$ be a Zariski open covering of $X$ splitting $\mathscr{L}$ and $\mathscr{V}$ via $\mathscr{O}_{U_i}$-module isomorphisms

$$\lambda_i : \mathscr{O}_{U_i} \xrightarrow{\sim} \mathscr{L}|_{U_i}, \quad \nu_i : \mathscr{O}_{U_i} \to \mathscr{V}|_{U_i},$$

for each $i \in I$, and similitudes,

$$\begin{array}{rcl} (\varphi_i, \lambda_i) : \mathscr{H}|_{U_i} = H_{\mathscr{O}_{U_i}}(\mathscr{O}_{U_i}) & \to & H_{\mathscr{L}}(\mathscr{V})|_{U_i} \\ e_1 & \mapsto & \nu_i e_1 \\ f_1 & \mapsto & \lambda_i f_1 \nu_i^{-1} \end{array}$$

where $e_1$ and $f_1$ are global sections of $\mathscr{H}$ forming a hyperbolic basis. Now define

$$l_{ij} = \lambda_{i\underline{j}}{}^{-1} \circ \lambda_{\underline{i}j} \in \mathbb{G}_{\mathrm{m}}(U_{ij}), \quad v_{ij} = \nu_{i\underline{j}}{}^{-1} \circ \nu_{\underline{i}j} \in \mathbb{G}_{\mathrm{m}}(U_{ij}),$$

and

$$a_{ij} = \varphi_{i\underline{j}}{}^{-1} \circ \varphi_{\underline{i}j} \in \mathbf{GO}_{1,1}(U_{ij}).$$

With respect to the ordered basis $e_1, f_1$, we express $a_{ij}$ as matrices,

$$a_{ij} = \begin{pmatrix} v_{ij} & 0 \\ 0 & l_{ij}\, v_{ij}{}^{-1} \end{pmatrix} = \tilde{l}_{ij} \begin{pmatrix} (\tilde{l}_{ij}\, v_{ij}{}^{-1})^{-1} & 0 \\ 0 & \tilde{l}_{ij}\, v_{ij}{}^{-1} \end{pmatrix},$$

refining over the étale topology, if necessary, the covering $\mathscr{U}$ so that over each $U_{ij}$ we can choose a square root $\tilde{l}_{ij}$ of $l_{ij}$. Similarly for the $v_{ij}$. Then over $U_{ij}$, one checks that $a_{ij}$ lift to sections

$$A_{ij} = \tilde{v}_{ij} \left( \tilde{l}_{ij}\, v_{ij}{}^{-1} + \frac{1}{2}(1 - \tilde{l}_{ij}\, v_{ij}{}^{-1}) e_1 f_1 \right) \in \mathbf{\Gamma}_{1,1}(U_{ij}).$$

Then an étale Čech 2-cocycle representing the coboundary map $gw_2^{\mathscr{H}}(H_{\mathscr{L}}(\mathscr{V})) \in H^2_{\text{ét}}(X, \boldsymbol{\kappa}_{1,1})$, is given by $A_{ij} A_{jk} A_{ik}{}^{-1}$. A straightforward (but tedious) computation in the Clifford algebra shows that

$$A_{ij} A_{jk} A_{ik}{}^{-1} \quad = \quad \frac{\tilde{v}_{ij}\, \tilde{v}_{jk}}{\tilde{v}_{ik}} \frac{\tilde{l}_{ij}\, \tilde{l}_{jk}}{\tilde{l}_{ik}} \left( 1 - \frac{1}{2} \left( 1 - \frac{\tilde{l}_{ik}}{\tilde{l}_{ij}\, \tilde{l}_{jk}} \right) e_1 f_1 \right) \in \boldsymbol{\kappa}_{1,1}(U_{ijk})$$

Note that the étale Čech $\boldsymbol{\mu}_2$-valued 2-cocycles,

$$\frac{\tilde{v}_{ij}\,\tilde{v}_{jk}}{\tilde{v}_{ik}} \quad \text{and} \quad \frac{\tilde{l}_{ij}\,\tilde{l}_{jk}}{\tilde{l}_{ik}},$$

are representatives of the classes $c_1(\mathscr{V}, \boldsymbol{\mu}_2)$ and $c_1(\mathscr{L}, \boldsymbol{\mu}_2)$ in $H^2_{\text{ét}}(X, \boldsymbol{\mu}_2)$, respectively.

Now for our fixed choice of global section $\epsilon = 1 - e_1 f_1$ of $\boldsymbol{\Gamma}_{1,1}$ satisfying

$$N(\epsilon) = -1, \quad r(\epsilon) = -\mathrm{id}, \quad \text{and} \quad \epsilon^2 = 1,$$

the étale Čech $\boldsymbol{\kappa}_{1,1}$-valued 2-cocycle,

$$1 - \frac{1}{2}\left(1 - \frac{\tilde{l}_{ik}}{\tilde{l}_{ij}\,\tilde{l}_{jk}}\right) e_1 f_1 = \begin{cases} 1 & \text{if } \frac{\tilde{l}_{ij}\,\tilde{l}_{jk}}{\tilde{l}_{ik}} = 1 \\ \epsilon|_{U_{ijk}} & \text{if } \frac{\tilde{l}_{ij}\,\tilde{l}_{jk}}{\tilde{l}_{ik}} = -1 \end{cases},$$

is trivial modulo $\boldsymbol{\kappa}^+_{1,1}$, i.e. modulo $\epsilon$. Finally, we have

$$p^+_* g w_2^{\mathscr{H}}(H_{\mathscr{L}}(\mathscr{V})) = c_1(\mathscr{V}, \boldsymbol{\mu}_2) + c_1(\mathscr{L}, \boldsymbol{\mu}_2),$$

in $H^2_{\text{ét}}(X, \boldsymbol{\mu}_2)$.

For the general case of $m$ odd, and since the formula is invariant under base change, use the splitting principle for Chern classes of vector bundles to reduce to the case where $\mathscr{V} \cong \mathscr{V}_1 \oplus \cdots \oplus \mathscr{V}_m$ is a direct sum of line bundles. Then choose, as before, a Zariski covering $\mathscr{U} = \{U_i\}_{i \in I}$, simultaneously trivializing $\mathscr{V}_1, \ldots, \mathscr{V}_m$ and $\mathscr{L}$ via $\mathscr{O}_{U_i}$-module isomorphisms,

$$\nu_i^s : \mathscr{O}_{U_i} \xrightarrow{\sim} \mathscr{V}_s|_{U_i}, \quad \lambda_i : \mathscr{O}_{U_i} \xrightarrow{\sim} \mathscr{L}|_{U_i},$$

for each $s = 1, \ldots, m$ and $i \in I$, and furthermore by refining the cover in the étale topology, ensuring that the transition maps,

$$v^s_{ij} \in \mathbb{G}_{\mathrm{m}}(U_{ij}), \quad l_{ij} \in \mathbb{G}_{\mathrm{m}}(U_{ij}),$$

all have square roots. Then for each $(i, j) \in I^2$ define trivializing similarities,

$$\begin{array}{rcl} (\varphi_i, \lambda_i) : \mathscr{H}|_{U_{ij}} & \rightarrow & H_{\mathscr{L}}(\mathscr{V})|_{U_i} \\ e_s & \mapsto & \nu_i^s\, e_s \\ f_s & \mapsto & \lambda_i\, f_s\, \nu_i^{s-1} \end{array}$$

where $e_1, \ldots, e_m, f_1, \ldots, f_m$ are global sections of $\mathscr{H}$ forming a hyperbolic basis. With respect to this basis, we express the transition maps,

$$a_{ij} = \varphi_{i\underline{j}}^{-1} \circ \varphi_{\underline{i}j} = \begin{pmatrix} v^1_{ij} & & & & & \\ & \ddots & & & & \\ & & v^m_{ij} & & & \\ & & & l_{ij}\, v^1_{ij} & & \\ & & & & \ddots & \\ & & & & & l_{ij}\, v^m_{ij} \end{pmatrix},$$

in $\mathbf{GO}_{m,m}(U_{ij})$. Then over $U_{ij}$, one can lift $a_{ij}$ to sections,

$$A_{ij} = \left(\frac{1}{l_{ij}}\right)^{\frac{m-1}{2}} \prod_{s=1}^{m} \tilde{v}_{ij}^{s} \left(\tilde{l}_{ij}\, v_{ij}^{s\;-1} + \frac{1}{2}\left(1 - \tilde{l}_{ij}\, v_{ij}^{s\;-1}\right) e_s f_s\right),$$

of $\mathbf{\Gamma}_{m,m}(U_{ij})$. A similar computation in the Clifford algebra yields

$$A_{ij}\, A_{jk}\, A_{ik}{}^{-1} = \left(\frac{\tilde{l}_{ij}\, \tilde{l}_{jk}}{\tilde{l}_{ik}}\right)^{\frac{m+1}{2}} \prod_{s=1}^{m} \frac{\tilde{v}_{ij}^{s}\, \tilde{v}_{jk}^{s}}{\tilde{v}_{ik}^{s}} \left(1 - \frac{1}{2}\left(1 - \frac{\tilde{l}_{ik}}{\tilde{l}_{ij}\, \tilde{l}_{jk}}\right) e_s f_s\right).$$

Note that the étale Čech $\boldsymbol{\mu}_2$-valued 2-cocycles,

$$\prod_{s=1}^{m} \frac{\tilde{v}_{ij}^{s}\, \tilde{v}_{jk}^{s}}{\tilde{v}_{ik}^{s}}, \qquad \left(\frac{\tilde{l}_{ij}\, \tilde{l}_{jk}}{\tilde{l}_{ik}}\right)^{\frac{m+1}{2}},$$

represent $c_1(\mathscr{V}, \boldsymbol{\mu}_2)$ and $\frac{m+1}{2} c_1(\mathscr{L}, \boldsymbol{\mu}_2)$ in $H^2_{\text{ét}}(X, \boldsymbol{\kappa}^0_{m,m}) = H^2_{\text{ét}}(X, \boldsymbol{\mu}_2)$, respectively, while the $\boldsymbol{\kappa}_{m,m}$-valued 2-cocycle,

$$\prod_{s=1}^{m} \left(1 - \frac{1}{2}\left(1 - \frac{\tilde{l}_{ik}}{\tilde{l}_{ij}\, \tilde{l}_{jk}}\right) e_s f_s\right) = \begin{cases} 1 & \text{if } \frac{\tilde{l}_{ij}\, \tilde{l}_{jk}}{\tilde{l}_{ik}} = 1 \\ \epsilon|_{U_{ijk}} & \text{if } \frac{\tilde{l}_{ij}\, \tilde{l}_{jk}}{\tilde{l}_{ik}} = -1 \end{cases},$$

is trivial modulo $\boldsymbol{\kappa}^+_{m,m}$, where

$$\epsilon = \prod_{s=1}^{m} (1 - e_s f_s) \in \mathbf{\Gamma}_{m,m}(X),$$

is our fixed global splitting section. Finally, we have

$$p^+_* gw_2^{\mathscr{H}}(H_{\mathscr{L}}(\mathscr{V})) = c_1(\mathscr{V}) + \frac{m+1}{2} c_1(\mathscr{L}),$$

in $H^2_{\text{ét}}(X, \boldsymbol{\mu}_2)$.

The case of $m$ even is similar, though we work over the étale double cover $\widetilde{X} = X \times_{\operatorname{Spec}\mathbb{Z}[\frac{1}{2}]} \operatorname{Spec}\mathbb{Z}[\frac{1}{2}, \sqrt{-1}] \to X$, where $\boldsymbol{\kappa}_{m,m} \cong \boldsymbol{\mu}_4$ is trivialized as the constant group scheme $\mathbb{Z}/4\mathbb{Z}$. $\square$

*Remark* 2.26. By the splitting principle for metabolic forms (see Theorem 1.29), we can reduce the calculation of $gw_2^{\mathscr{H}}$ of an oriented metabolic form to the above calculation for hyperbolic forms.

# Chapter 3

# Forms of low rank

The classification of quadratic forms of low rank is an endeavor going back to the third century Greek geometers, who classified quadratic forms of rank 2 and 3 over the real number via the geometry of conic sections. Over fields, Minkowski first gave the classification of quadratic forms (of arbitrary rank) over $\mathbb{Q}$ in 1890. Dickson gave the classification over finite fields $\mathbb{F}_p$ in 1899 and attacked the question over general fields in 1907. It was Witt's famous 1937 paper that, for the first time, truly used the power of algebra and geometry to bring the classification problem to a modern setting. The next breakthrough was Pfister's series of papers in the 1960s.

In the context of quadratic forms over schemes, *low rank* usually means of rank $\leq 6$. In this interval, the accidental isomorphisms of Dynkin diagrams, $\mathsf{A}_1 = \mathsf{B}_1 = \mathsf{C}_1$, $\mathsf{D}_2 = \mathsf{A}_1 + \mathsf{A}_1$, $\mathsf{B}_2 = \mathsf{C}_2$, and $\mathsf{A}_3 = \mathsf{D}_3$, have beautiful reverberations in the theory of quadratic forms of rank 3, 4, 5, and 6, respectively. Aside from connections with triality for forms of rank 7 and 8, beyond this interval, there are no further accidental isomorphisms to exploit, and the theory becomes much more difficult. Over rings, the accidental isomorphisms were heavily utilized by Knus, Ojanguren, Parimala, Paques, and Sridharan in the 1980s and 1990s. Now, a standard reference on this work is Knus [34, Chapter V]. Over fields (and more generally, over division algebras), a wonderful reference is Knus/Merkurjev/Rost/Tignol [35, IV §15]. Over general schemes, much of the theory over rings can be globalized, but a unified treatment does not yet exist in the literature.

Concerning the classification of line bundle-valued quadratic forms of low rank, when the line bundle is a square in the Picard group, the theory reduces to that of classical $\mathscr{O}_X$-valued symmetric bilinear forms over schemes. In particular, this applies to schemes with trivial Picard group (e.g. local rings), and to the case of odd rank forms. As for the case when the line bundle is not a square, Bichsel/Knus [7] mostly handle the case of rank 4 and 6 with trivial Arf invariant over affine schemes. Here, to give a taste of the theory, we'll provide a treatment of the classification of line bundle-valued symmetric bilinear forms over schemes (with 2 invertible) of rank 6 from the point of view of cohomological invariants.

As usual, let $X$ be a noetherian scheme with $\frac{1}{2} \in \mathscr{O}_X$, considered in the étale topology, and let $\mathscr{L}$ be a fixed line bundle (i.e. invertible $\mathscr{O}_X$-module) on $X$. As usual, for simplicity we will assume that $X$ is connected.

## 3.1 Forms of rank 6

The "accidental isomorphism" of the Dynkin diagrams $\mathsf{A}_3$ and $\mathsf{D}_3$ is reflected in the isomorphism of algebraic groups $\mathbf{SL}_4 \cong \mathbf{Spin}(\mathscr{A}lt_4(\mathscr{O}_X), \mathrm{pf})$, where $(\mathscr{A}lt_4(\mathscr{O}_X), \mathrm{pf})$ is the canonical quadratic form given by the pfaffian map on the rank 6 space of alternating $4{\times}4$ matrices. There's a dictionary between certain degree 4 Azumaya algebras of period 2 and symmetric bilinear forms of rank 6 with trivial Arf invariant, via the reduced pfaffian construction of Knus [31] and Knus/Parimala/Sridharan [32]. Later, Bichsel/Knus [7] gave an extension of this work to the line bundle-valued case.

For a construction of the reduced pfaffian over any base scheme using Brauer-Severi varieties, see Parimala/Sridharan [40]. In this section, we use an alternate construction of the reduced pfaffian over any base scheme to clarify and complete the above dictionary. We show that there's an equivalence of categories between 2-torsion data (i.e. equivalence classes of degree 4 Azumaya algebras with chosen trivializations of their tensor squares) and oriented line bundle-valued symmetric bilinear forms of rank 6 with trivial Arf invariant. This dictionary is reflected in the isomorphism of algebraic groups $\mathbf{GL}_4/\boldsymbol{\mu}_2 \cong \mathbf{GSO}(\mathscr{A}lt_4(\mathscr{O}_X), \mathrm{pf})$. The main result of this section is to show that the similarity 2nd Hasse-Witt invariant respects this refined equivalence of categories.

### 3.1.1 2-torsion datum

Consider the sheaf of groups $\mathbf{GL}_n/\boldsymbol{\mu}_2$ defined as the sheaf cokernel of the exact sequence of sheaves of groups,

$$1 \to \boldsymbol{\mu}_2 \to \mathbf{GL}_n \to \mathbf{GL}_n/\boldsymbol{\mu}_2 \to 1, \tag{3.1}$$

in the étale topology on $X$. Then $\mathbf{GL}_n/\boldsymbol{\mu}_2$ is a smooth affine algebraic group on $X$. Note that $\mathbf{GL}_1/\boldsymbol{\mu}_2 \cong \mathbb{G}_\mathrm{m}$ via the Kummer sequence, but that in general, $\mathbf{GL}_n/\boldsymbol{\mu}_2$ is not isomorphic to $\mathbf{GL}_n$ for $n \geq 2$. The category of $\mathbf{GL}_n/\boldsymbol{\mu}_2$-torsors on $X$ is equivalent to the category of *2-torsion data*, see Knus [34, III §9.3].

**Definition 3.1.** A *2-torsion datum* on $X$ is a triple $(\mathscr{A}, \mathscr{P}, \varphi)$, consisting of an Azumaya algebra $\mathscr{A}$ of rank $n^2$ on $X$, a vector bundle $\mathscr{P}$ of rank $n^2$ on $X$, and an $\mathscr{O}_X$-algebra isomorphism $\varphi : \mathscr{A} \otimes \mathscr{A} \xrightarrow{\sim} \mathscr{E}nd(\mathscr{P})$. In particular, the class of $\mathscr{A}$ in the Brauer group has period $\leq 2$. We will call $n$ the *degree* of the 2-torsion datum. A isomorphism of 2-torsion data is a pair $(\psi, g) : (\mathscr{A}, \mathscr{P}, \varphi) \to (\mathscr{A}', \mathscr{P}', \varphi')$, where $\psi : \mathscr{A} \to \mathscr{A}'$ is an $\mathscr{O}_X$-algebra isomorphism and $g : \mathscr{P} \to \mathscr{P}'$ is an $\mathscr{O}_X$-module isomorphism rendering commutative the following diagram of $\mathscr{O}_X$-algebras,

$$\begin{tikzcd} \mathscr{A} \otimes \mathscr{A} \arrow[r, "\varphi"] \arrow[d, "\psi \otimes \psi"'] & \mathscr{E}nd(\mathscr{P}) \arrow[d, "i_g"] \\ \mathscr{A}' \otimes \mathscr{A}' \arrow[r, "\varphi'"] & \mathscr{E}nd(\mathscr{P}') \end{tikzcd}$$

where $i_g$ is defined locally on sections over $U \to X$ by $f \mapsto g \circ f \circ g^{-1}$. The category of 2-torsion datum on $X$ is thus a groupoid.

To every vector bundle $\mathscr{V}$ of rank $n$ on $X$, we associate a *split datum* $(\mathscr{E}nd(\mathscr{V}), \mathscr{V} \otimes \mathscr{V}, \varphi_{\mathscr{V}})$, where

$$\varphi_{\mathscr{V}} : \mathscr{E}nd(\mathscr{V}) \otimes \mathscr{E}nd(\mathscr{V}) \xrightarrow{\sim} \mathscr{E}nd(\mathscr{V} \otimes \mathscr{V}),$$

is the canonical $\mathscr{O}_X$-algebra isomorphism.

**Proposition 3.2.** *Let $X$ be a noetherian scheme with $\frac{1}{2} \in \mathscr{O}_X$. The category of 2-torsion data of degree $n$ on $X$ is equivalent to the category of $\mathbf{GL}_n/\boldsymbol{\mu}_2$-torsors for the étale topology.*

*Proof.* As in Appendix A.2.2, it's enough to verify that every 2-torsion datum of degree $n$ is locally isomorphic (for the étale topology) to a split datum (this is done in Knus [34, III Lemma 9.3.1]), that every $\mathbf{GL}_n/\boldsymbol{\mu}_2$-torsor has the structure of a 2-torsion datum of degree $n$, and then to show that the isomorphism group scheme of a certain split datum is isomorphic to $\mathbf{GL}_n/\boldsymbol{\mu}_2$. To this end, let $\mathscr{T} = (\mathscr{E}nd(\mathscr{O}_X^n), \mathscr{O}_X^n \otimes \mathscr{O}_X^n, \varphi_{\mathscr{O}_X^n})$ be the split datum associated to the trivial vector bundle of rank $n$ on $X$.

We proceed following Knus/Merkujev/Rost/Tignol [35, VII §31 Exercise 5]. For any 2-torsion datum $(\mathscr{A}, \mathscr{P}, \varphi)$ of rank $n$, the conditions defining automorphisms of 2-torsion data translate into a cartesian diagram of group schemes

$$\begin{tikzcd}
\mathbf{Aut}(\mathscr{A}, \mathscr{P}, \varphi) \arrow[rr] \arrow[d] & & \mathbf{GL}(\mathscr{P}) \arrow[d, "i"] \\
\mathbf{Aut}_{\mathscr{O}_X\text{-alg}}(\mathscr{A}) \arrow[rr, "\varphi \circ (-)^{\otimes 2} \circ \varphi^{-1}"] & & \mathbf{GL}(\mathscr{P}).
\end{tikzcd}$$

Thus on sections over $U \to X$ we have

$$\mathbf{Aut}(\mathscr{A}, \mathscr{P}, \varphi)(U) = \{(\psi, g) \in \mathbf{Aut}_{\mathscr{O}_X\text{-alg}}(\mathscr{A})(U) \times \mathbf{GL}(\mathscr{P})(U) : \varphi \circ (\psi \otimes \psi) = i_g \circ \varphi\}.$$

Now identifying $\mathbf{PGL}_{n^2} = \mathbf{Aut}_{\mathscr{O}_X-\mathrm{alg}}(\mathscr{E}nd(\mathscr{O}_X^n))$ and $\mathbf{GL}_{n^2} = \mathbf{GL}(\mathscr{O}_X^n \otimes \mathscr{O}_X^n)$, we have

$$\mathbf{Aut}(\mathscr{T})(U) = \{(\psi, g) \in \mathbf{PGL}_{n^2}(U) \times \mathbf{GL}_{n^2}(U) : \varphi_{\mathscr{O}_U^n} \circ (\psi \otimes \psi) = i_g \circ \varphi_{\mathscr{O}_U^n}\}.$$

As for identifying $\mathrm{Aut}(\mathscr{T})$, we'll show that the homomorphism

$$\begin{array}{rcl} \mathbf{GL}_n(U) & \to & \mathbf{Aut}(\mathscr{T})(U) \\ \alpha & \mapsto & (i_\alpha, \alpha \otimes \alpha) \end{array}$$

is locally surjective in the étale topology on $X$, and has kernel $\boldsymbol{\mu}_2 \hookrightarrow \mathbf{GL}_n$. To this end, let $(\psi, g) \in \mathbf{Aut}(\mathscr{T})(U)$. By the strong Skolem-Noether theorem, see Milne [36, IV Proposition 2.3], there's an exact sequence of group schemes,

$$1 \to \mathbb{G}_\mathrm{m} \to \mathbf{GL}_n \xrightarrow{i} \mathbf{Aut}_{\mathscr{O}_X\text{-alg}}(\mathscr{E}nd(\mathscr{O}_X^n)) \to 1.$$

in the étale topology on $X$. In particular, there exists an étale map $V \to U$ and a section $\beta \in \mathbf{GL}_n(V)$ so that $i_\beta = \psi|_V$. But now by the cartesian product structure of $\mathbf{Aut}(\mathscr{T})$ and a general formula (the final equality), we have

$$i_{g|_V} = \varphi_{\mathscr{O}_V^n} \circ \psi|_V \otimes \psi|_V \circ \varphi_{\mathscr{O}_V^n}^{-1} = \varphi_{\mathscr{O}_V^n} \circ i_\beta \otimes i_\beta \circ \varphi_{\mathscr{O}_V^n}^{-1} = i_{\beta \otimes \beta}.$$

Again using the strong Skolem-Noether theorem, this time for the group $\mathbf{GL}_{n^2}$, we can find an étale map $V' \to V$ and a section $c \in \mathbb{G}_\mathrm{m}(V')$, so that $g|_{V'} = c \cdot \beta|_{V'} \otimes \beta|_{V'}$. Now let refining $V'$ to an étale $V'' \to V$ if necessary, we can assume that $c$ has a square root $\tilde{c}$. Finally, letting $\alpha = \tilde{c} \cdot \alpha|_{V''} \in \mathbf{GL}_n(V'')$, we have that $(i_\alpha, \alpha \otimes \alpha) = (\psi|_{V''}, g|_{V''})$. Thus the homomorphism $GL_n \to \mathbf{Aut}(\mathscr{T})$ is locally surjective in the étale topology, thus is an epimorphism of group schemes. Its kernel is clearly the diagonal subgroup $\boldsymbol{\mu}_2 \hookrightarrow \mathbf{GL}_n$. Thus we've identified $\mathbf{Aut}(\mathscr{T})$ with the sheaf cokernel of $\boldsymbol{\mu}_2 \to \mathbf{GL}_n$, i.e. with $\mathbf{GL}_n/\boldsymbol{\mu}_2$. □

The interpretation of exact sequence (3.1) on nonabelian étale Čech cohomology,

$$H^1_{\text{ét}}(X, \mathbf{GL}_n) \to H^1_{\text{ét}}(X, \mathbf{GL}_n/\boldsymbol{\mu}_2) \to H^2_{\text{ét}}(X, \boldsymbol{\mu}_2) \tag{3.2}$$

is that the isomorphism class of a vector bundle $\mathscr{V}$ of rank $n$ is mapped to the isomorphism class of the associated split datum $(\mathscr{E}nd(\mathscr{V}), \mathscr{V} \otimes \mathscr{V}, \varphi_{\mathscr{V}})$. The 2nd coboundary map, assigning an isomorphism class of 2-torsion datum $(\mathscr{A}, \mathscr{P}, \varphi)$ to a cohomology class $a(\mathscr{A}, \mathscr{P}, \varphi) \in H^2_{\text{ét}}(X, \boldsymbol{\mu}_2)$ is related to the *involutive Brauer group*.

### 3.1.2 The involutive Brauer group

An *involution* (of the first kind) on an $\mathscr{O}_X$-algebra $\mathscr{A}$ is an $\mathscr{O}_X$-algebra isomorphism $\sigma : \mathscr{A} \to \mathscr{A}^{\text{op}}$ satisfying $\sigma^{\text{op}} \circ \sigma = \text{id}$. We will consider Azumaya algebras $\mathscr{A}$ on $X$ with involution, see §1.1.1. Locally in the étale topology, see Milne [36, IV Proposition 2.1], $\mathscr{A}$ is isomorphic to an endomorphism algebra, and the involution restricts to an involution on an endomorphism algebra, see Proposition 1.2 or Knus/Parimala/Srinivas [33]. Such involutions are either of *orthogonal type* or *symplectic type*. For a given Azumaya algebra with involution, the type is locally constant in the étale topology on $X$ and gives rise to a global section of $\boldsymbol{\mu}_2$. We will only be interested in Azumaya algebras with orthogonal involutions.

Parimala/Srinivas [41] construct a "Brauer group" consisting of isomorphism classes of Azumaya algebras with involution (of any type) modulo classes of endomorphism algebras with involution coming from $\mathscr{O}_X$-valued symmetric bilinear forms on $X$. We will denote by $\mathrm{Br}^+(X)$ the subgroup consisting of classes of Azumaya algebras with orthogonal involutions.

The category of Azumaya algebras of degree $n$ with orthogonal involution is equivalent to the category of $\mathbf{PO}_n \cong \mathbf{PGO}_n$-torsors on $X$, see Theorem 1.4. As in §1.1.1, to every $\mathscr{O}_X$-valued symmetric bilinear space $(\mathscr{E}, b)$ of rank $n$ on $X$, we associate an Azumaya algebra $(\mathscr{E}nd(\mathscr{E}), \sigma_b)$ with orthogonal involution, where $\sigma(\varphi) = \psi_b^{-1} \circ \varphi^\vee \circ \psi_b$ on sections over $U \to X$. The interpretation of the projective orthogonal exact sequence,

$$1 \to \boldsymbol{\mu}_2 \to \mathbf{O}_n \to \mathbf{PO}_n \to 1$$

on nonabelian étale Čech cohomology,

$$H^1_{\text{ét}}(X, \mathbf{O}_n) \to H^1_{\text{ét}}(X, \mathbf{PO}_n) \to H^2_{\text{ét}}(X, \boldsymbol{\mu}_2),$$

is that the isometry class of an $\mathscr{O}_X$-valued symmetric bilinear space of rank $n$ on $X$ is mapped to the isomorphism class of its associated endomorphism algebra with involution. The 2nd coboundary map is trivial on classes of such endomorphism algebras with involution, and thus factors through $\mathrm{Br}^+(X)$, yielding a group homomorphism

$$a : \mathrm{Br}^+(X) \to H^2_{\text{ét}}(X, \boldsymbol{\mu}_2).$$

By Parimala/Srinivas [41, Theorem 1] this homomorphism is injective, and is surjective if and only if every 2-torsion element in $H^2_{\text{ét}}(X, \mathbb{G}_{\text{m}})$ is represented by an Azumaya algebra on $X$ (for example, this holds for any affine scheme).

There's a commutative diagram of "forgetful" homomorphisms of group schemes in the étale topology on $X$, with exact rows (but not exact columns),

$$\begin{tikzcd}
1 \arrow[r] & \boldsymbol{\mu}_2 \arrow[r] \arrow[d, equal] & \mathbf{O}_n \arrow[r] \arrow[d] & \mathbf{PO}_n \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \boldsymbol{\mu}_2 \arrow[r] \arrow[d] & \mathbf{GL}_n \arrow[r] \arrow[d, equal] & \mathbf{GL}_n/\boldsymbol{\mu}_2 \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbb{G}_{\mathrm{m}} \arrow[r] & \mathbf{GL}_n \arrow[r] & \mathbf{PGL}_n \arrow[r] & 1
\end{tikzcd}$$

The map $H^1_{\text{ét}}(X, \mathbf{PO}_n) \to H^1_{\text{ét}}(X, \mathbf{GL}_n/\boldsymbol{\mu}_2)$ is interpreted as associating to an Azumaya algebra $(\mathscr{A}, \sigma)$ of degree $n$ with an orthogonal involution, the canonical 2-torsion datum $(\mathscr{A}, \mathscr{A}, \varphi_\sigma)$ given on sections by

$$\begin{array}{rcccl} \varphi_\sigma & : & \mathscr{A} \otimes \mathscr{A} & \xrightarrow{\sim} & \mathscr{E}nd(\mathscr{A}) \\ & & a \otimes b & \longmapsto & x \mapsto ax\sigma(b). \end{array}$$

The map $H^1_{\text{ét}}(X, \mathbf{GL}_n/\boldsymbol{\mu}_2) \to H^1_{\text{ét}}(X, \mathbf{PGL}_n)$ is interpreted as associating to a 2-torsion datum its underlying Azumaya algebra. The cohomological coboundary,

$$H^1_{\text{ét}}(X, \mathbf{PO}_n) \to H^2_{\text{ét}}(X, \boldsymbol{\mu}_2),$$

associates to an Azumaya algebra $(\mathscr{A}, \sigma)$ with orthogonal involution, the image of its class in the involutive Brauer group under the injective homomorphism $a$ described above. There's also a commutative diagram,

$$\begin{tikzcd}
H^1_{\text{ét}}(X, \mathbf{PO}_n) \arrow[r] \arrow[d] & \mathrm{Br}^+(X) \arrow[r, hook, "a"] \arrow[d] & H^2_{\text{ét}}(X, \boldsymbol{\mu}_2) \arrow[d] \\
H^1_{\text{ét}}(X, \mathbf{PGL}_n) \arrow[r] & \mathrm{Br}(X) \arrow[r, hook] & H^2_{\text{ét}}(X, \mathbb{G}_{\mathrm{m}})
\end{tikzcd}$$

showing that the invariant $a(\mathscr{A}, \sigma)$ is a refinement of the class of $\mathscr{A}$ in the Brauer group.

By a result of Saltman [45] that generalizes a theorem of Albert, for every 2-torsion Azumaya algebra $\mathscr{A}$, there's an Azumaya algebra $(\mathscr{A}', \sigma)$ with orthogonal involution such that $\mathscr{A}'$ Brauer equivalent to $\mathscr{A}$. Indeed, considering the following commutative diagram,

$$\begin{tikzcd}
1 \arrow[r] & \boldsymbol{\mu}_2 \arrow[r] \arrow[d, equal] & \mathbf{GL}_n \arrow[r] \arrow[d] & \mathbf{GL}_n/\boldsymbol{\mu}_2 \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \boldsymbol{\mu}_2 \arrow[r] & \mathbf{O}_{n,n} \arrow[r] & \mathbf{PO}_{n,n} \arrow[r] & 1
\end{tikzcd}$$

of group schemes in the étale topology on $X$ with exact rows, the map $H^1_{\text{ét}}(X, \mathbf{GL}_n/\boldsymbol{\mu}_2) \to H^1_{\text{ét}}(X, \mathbf{PO}_{n,n})$ is interpreted as the following construction of Knus/Parimala/Srinivas [33]. Given a 2-torsion datum $(\mathscr{A}, \mathscr{P}, \varphi)$ of degree $n$, the authors construct a degree $2n$ Azumaya algebra $(\mathscr{A}', \sigma_\varphi)$ with orthogonal involution, where $\mathscr{A}' = (\mathscr{E}nd_{\mathscr{A}^{\mathrm{op}}}(\mathscr{A}^{\mathrm{op}} \oplus \mathscr{P}), \sigma_\varphi)$. By the commutativity of the diagram, the 2nd coboundary maps coincide $a(\mathscr{A}', \sigma_\varphi) = a(\mathscr{A}, \mathscr{P}, \varphi)$ in $H^2_{\text{ét}}(X, \boldsymbol{\mu}_2)$. By abuse of notation, if $(\mathscr{A}, \mathscr{P}, \varphi)$ is a 2-torsion datum, we simply refer to its class in $\mathrm{Br}^+(X)$ as the class of $(\mathscr{A}', \sigma_\varphi)$ just constructed.

### 3.1.3 The reduced pfaffian

Given a 2-torsion datum $(\mathscr{A}, \mathscr{P}, \varphi)$ of degree $n$ on $X$, by the strong Skolem-Noether theorem the switch map $\omega_{\mathscr{A}} : \mathscr{A} \otimes \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$ is an inner automorphism. In fact, there's a canonical section $u \in \mathscr{A} \otimes \mathscr{A}(X)$ called the *Goldman element* satisfying $u^2 = \mathbf{1}_{\mathscr{A}}$, $i_u = \omega_{\mathscr{A}}$, and

$$\begin{array}{rcl} \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A} & \xrightarrow{\sim} & \mathscr{E}nd(\mathscr{A}) \\ u & \longmapsto & x \mapsto \mathrm{Trd}(x) \cdot \mathbf{1}_{\mathscr{A}}, \end{array}$$

by taking global sections via the canonical isomorphism, see Knus [34, Lemma 8.4.1]. Let $\psi = \varphi(u) : \mathscr{P} \to \mathscr{P}$ be the induced morphism of $\mathscr{O}_X$-modules. Then $\psi$ is an $\mathscr{A} - \mathscr{A}^{\mathrm{op}}$-algebra homomorphism satisfying $\psi^2 = \mathrm{id}_{\mathscr{P}}$ and is called the *module involution* of $(\mathscr{A}, \mathscr{P}, \varphi)$. Denote by $\mathscr{S}_{\varphi}^{-}(\mathscr{P})$ the sheaf image of the morphism $\mathrm{id}_{\mathscr{P}} - \psi$, i.e. the sheaf of *alternating elements* of $\mathscr{P}$ with respect to $\psi$. Then $\mathscr{S}_{\varphi}^{-}(\mathscr{P})$ is a locally free $\mathscr{O}_X$-module of rank $n(2n-1)$ on $X$, a fact that can be verified locally in the étale topology, see Knus [34, Theorem 9.3.2].

The reduced pfaffian construction associates to a 2-torsion datum $(\mathscr{A}, \mathscr{P}, \varphi)$ of degree $n$ on $X$, a functorial *pfaffian line bundle* $\mathrm{pf}(\mathscr{P})$ and polynomial map $\mathrm{pf} : \mathscr{S}_{\varphi}^{-}(\mathscr{P}) \to \mathrm{pf}(\mathscr{P})$ of degree $n$. Applied to a split datum $(\mathscr{E}nd(\mathscr{V}), \mathscr{V} \otimes \mathscr{V}, \varphi_{\mathscr{V}})$, the module involution $\psi : \mathscr{V} \otimes \mathscr{V} \to \mathscr{V} \otimes \mathscr{V}$ coincides with the switch map and $\mathscr{S}_{\varphi}^{-}(\mathscr{V} \otimes \mathscr{V}) = \bigwedge^2 V$, then the reduced pfaffian gives $\mathrm{pf}(\mathscr{V} \otimes \mathscr{V}) = \det \mathscr{V}$ and $\mathrm{pf} : \bigwedge^2 \mathscr{V} \to \det \mathscr{V}$ coincides with the classical pfaffian. Knus [31] and Knus/Parimala/Sridharan [32] construct the reduced pfaffian over affine schemes by faithfully flat descent. Parimala/Sridharan [40], give a construction over arbitrary schemes by pulling back to Brauer-Severi varieties. Over schemes in which 2 is invertible, we realize the reduced pfaffian, restricted to 2-torsion data of degree 4, as a map on torsors induced from an exceptional isomorphism of algebraic group schemes.

Applied to a 2-torsion datum of degree 4, the reduced pfaffian yields a symmetric bilinear space of rank 6 with valued in the pfaffian line bundle and with trivial Arf invariant. This fact was the basis for much work on the classification of rank 6 symmetric bilinear spaces by degree 4 Azumaya algebras, see Knus [31], Knus/Parimala/Sridharan [32], and Bichsel/Knus [7]. Our realization of the reduced pfaffian in this case clarifies and organizes these results.

Let $\mathscr{A}lt_4 = \mathscr{A}lt_4(\mathscr{O}_X)$ be the free vector subbundle of $\mathscr{E}nd(\mathscr{O}_X^4)$ consisting of alternating matrices and $\mathrm{pf} : \mathscr{A}lt_4 \to \mathscr{O}_X$ be the classical pfaffian map. Then $(\mathscr{A}lt_4, \mathrm{pf}, \mathscr{O}_X)$ is the reduced pfaffian construction applied to the 2-torsion datum $(\mathscr{E}nd(\mathscr{O}_X^4), \mathscr{E}nd(\mathscr{O}_X^4), \varphi)$ given by

$$\begin{array}{rcl} \varphi \; : \; \mathscr{E}nd(\mathscr{O}_X^4) \otimes \mathscr{E}nd(\mathscr{O}_X^4) & \xrightarrow{\sim} & \mathscr{E}nd(\mathscr{E}nd(\mathscr{O}_X^4)) \\ x \otimes y & \longmapsto & z \mapsto xyz^t \end{array}$$

Note that this 2-torsion datum is isomorphic to the split datum $(\mathscr{E}nd(\mathscr{O}_X^4), \mathscr{O}_X^4 \otimes \mathscr{O}_X^4, \varphi_{\mathscr{O}_X^4})$. Thus $(\mathscr{A}lt_4, \mathrm{pf})$ is an $\mathscr{O}_X$-valued symmetric bilinear form on $X$.

Finally, define a group scheme homomorphism by

$$\begin{array}{rcl} \mathbf{GL}_4 & \xrightarrow{\Phi} & \mathbf{GSO}(\mathscr{A}lt_4) \\ x & \longmapsto & y \mapsto xyx^t \end{array}$$

on sections, noting that by properties of the classical pfaffian $\mathrm{pf}(xyx^t) = \det(x)\,\mathrm{pf}(y)$ so that indeed $\Phi$ maps $x$ to a well-defined similarity transformation of $(\mathscr{A}lt_4, \mathrm{pf})$ with multiplier $\det(x)$.

It's also well-known that $\det(y \mapsto xyx^t) = \det(x)^3$, i.e. $\Phi$ maps $x$ to a proper similitude. In fact, by descent, given a 2-torsion datum $(\mathscr{A}, \mathscr{P}, \varphi)$ there's a canonically defined orientation

$$\zeta_{\mathscr{P}} : \langle -1 \rangle \xrightarrow{\sim} \operatorname{disc}(\mathscr{S}^-_\varphi(\mathscr{P}), \mathrm{pf}, \mathrm{pf}(\mathscr{P}))$$

on the reduced pfaffian.

**Theorem 3.3.** *Let $X$ be a noetherian scheme with $\frac{1}{2} \in \mathscr{O}_X$ and considered in the étale topology.*

*a) There's a commutative diagram with exact rows and columns,*

$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \boldsymbol{\mu}_2 \arrow[r] \arrow[d, equal] & \mathbf{SL}_4 \arrow[r, "\Phi"] \arrow[d] & \mathbf{SO}(\mathscr{A}lt_4) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \boldsymbol{\mu}_2 \arrow[r] & \mathbf{GL}_4 \arrow[r, "\Phi"] \arrow[d, "\det"'] & \mathbf{GSO}(\mathscr{A}lt_4) \arrow[r] \arrow[d, "\mu"] & 1 \\
 & & \mathbb{G}_{\mathrm{m}} \arrow[r, equal] \arrow[d] & \mathbb{G}_{\mathrm{m}} \arrow[d] & \\
 & & 1 & 1 &
\end{tikzcd}$$

*of group schemes in the étale topology on $X$.*

*b) The associated isomorphism of group schemes,*

$$\mathbf{GL}_4/\boldsymbol{\mu}_2 \xrightarrow{\sim} \mathbf{GSO}(\mathscr{A}lt_4)$$

*in the étale topology on $X$ yields an equivalence between the category of 2-torsion data of degree 4 on $X$ and the category of oriented similarity classes of line bundle-valued symmetric bilinear spaces of rank 6 with trivial Arf invariant on $X$. Furthermore, this isomorphism induces the (oriented) reduced pfaffian construction,*

$$\begin{array}{rcl} H^1_{\text{ét}}(X, \mathbf{GL}_4/\boldsymbol{\mu}_2) & \xrightarrow{\sim} & H^1_{\text{ét}}(X, \mathbf{GSO}(\mathscr{A}lt_4)) \\ (\mathscr{A}, \mathscr{P}, \varphi) & \longmapsto & (\mathscr{S}^-_\varphi(\mathscr{P}), \mathrm{pf}, \mathrm{pf}(\mathscr{P}), \zeta_{\mathscr{P}}) \end{array}$$

*on isomorphism classes of objects.*

*Remark* 3.4. The category of $\mathbf{SL}_4/\boldsymbol{\mu}_2$)-torsors is equivalent to the category of 2-torsion data with a fixed trivialization of the pfaffian line bundle. The isomorphism of group schemes $\mathbf{SL}_4/\boldsymbol{\mu}_2 \xrightarrow{\sim} \mathbf{SO}(\mathscr{A}lt_4)$ from Theorem 3.3 then yields, on isomorphism classes of torsors, exactly the classification of $\mathscr{O}_X$-valued symmetric bilinear forms of rank 6 with trivial Arf invariant obtained by Knus/Parimala/Sridharan [32].

We must point out that there are two points where our classification of rank 6 forms seems to differ from the approach of Knus [31], Knus/Parimala/Sridharan [32], and Bichsel/Knus [7]. In previous formulations of the classification of rank 6 forms, the complications of "similarity up to multiplication by a discriminant module" and "equivalence of Azumaya algebras" seemed to arise. We'll now explain how our classification circumvents these complications.

While the isomorphism class of the Azumaya algebra $\mathscr{A}$ doesn't uniquely determine the similarity class of the reduced pfaffian $(\mathscr{S}^-_\varphi(\mathscr{P}), \mathrm{pf}, \mathrm{pf}(\mathscr{P}))$ – it only determines the reduced pfaffian up to multiplication by an $\mathscr{O}_X$-valued line – the full 2-torsion datum $(\mathscr{A}, \mathscr{P}, \varphi)$ does uniquely determine the similarity class of the reduced pfaffian. Classically, the inverse functor of the reduced pfaffian was "half" the even Clifford algebra, or "half" the generalized even Clifford algebra in the line bundle-valued case. The even Clifford algebra is invariant under multiplication by $\mathscr{O}_X$-valued lines $(\mathscr{N}, n)$,

$$\begin{array}{rcl} \mathscr{C}_0(\mathscr{N} \otimes \mathscr{E}, n \otimes b, \mathscr{L}) & \xrightarrow{\sim} & \mathscr{C}_0(\mathscr{E}, b, \mathscr{L}) \\ (a \otimes x)\,(b \otimes y) & \longmapsto & n(a, b)\, xy \end{array}$$

while the Clifford bimodule is not,

$$\mathscr{C}_1(\mathscr{N} \otimes \mathscr{E}, n \otimes b, \mathscr{L}) \xrightarrow{\sim} \mathscr{N} \otimes \mathscr{C}_1(\mathscr{E}, b, \mathscr{L})$$

Thus the associated 2-torsion data differ between forms scaled by $\mathscr{O}_X$-lines. Specifically, if $\mathscr{N}$ is not a square in the Picard group, then

$$(\mathscr{C}_0(\mathscr{N} \otimes \mathscr{E}, n \otimes b, \mathscr{L}), \mathscr{C}_1(\mathscr{N} \otimes \mathscr{E}, n \otimes b, \mathscr{L}), \cdot) \cong (\mathscr{C}_0(\mathscr{E}, b, \mathscr{L}), \mathscr{N} \otimes \mathscr{C}_1(\mathscr{E}, b, \mathscr{L}), \mathrm{can}_{\mathscr{N}} \circ \cdot)$$

and

$$(\mathscr{C}_0(\mathscr{E}, b, \mathscr{L}), \mathscr{C}_1(\mathscr{E}, b, \mathscr{L}), \cdot)$$

define nonisomorphic 2-torsion data. Keeping track of the full 2-torsion data, not just the isomorphism class of the Azumaya algebra, removes the complication of considering forms up to multiplication by $\mathscr{O}_X$-lines.

The next complication arises because opposite Azumaya algebras yield similar reduced pfaffians. This led previous authors to introduce the notion of equivalence of Azumaya algebras $\mathscr{A} \sim \mathscr{A}'$ if $\mathscr{A} \cong \mathscr{A}'$ or $\mathscr{A}^{\mathrm{op}} \cong \mathscr{A}'$ (when the base scheme $X$ is connected, as we are always assuming). Again, keeping track of the full 2-torsion datum, not just the isomorphism class of the Azumaya algebra, eliminates this complication. Indeed, given a 2-torsion datum $(\mathscr{A}, \mathscr{P}, \varphi)$ the canonical opposite 2-torsion datum $(\mathscr{A}^{\mathrm{op}}, \mathscr{P}^\vee, \varphi^\vee)$ gives rise to a similar reduced pfaffian. However, the canonical orientation induced by the dual datum $\zeta_{\mathscr{P}^\vee}$ is equal to the orientation induced by the datum $\zeta_{\mathscr{P}}$ if and only if $\mathscr{A} \cong \mathscr{A}^{\mathrm{op}}$. Thus the oriented reduced pfaffian distinguishes between an Azumaya algebra and its dual (unless they are isomorphic).

### 3.1.4 Hasse-Witt invariants and the reduced pfaffian

Fix $(\mathscr{H}, h)$ as the $\mathscr{O}_X$-valued symmetric bilinear space associated to $(\mathscr{A}lt_4, \mathrm{pf})$. The choice of $\mathscr{A}lt_3 \subset \mathscr{A}lt_4$ as a lagrangian induces an isometry $(\mathscr{H}, h) \xrightarrow{\sim} H_{\mathscr{O}_X}(\mathscr{O}_X^3)$, under which the global section

$$\epsilon = (1 - 2e_{12}e_{34})(1 + 2e_{13}e_{24})(1 - 2e_{23}e_{14}) \in \boldsymbol{\kappa}(X),$$

(where the global sections $e_{ij}$ of $\mathscr{A}lt_4(\mathscr{O}_X)$ form the standard basis of the alternating matrices) maps to the fixed global section $\epsilon \in \boldsymbol{\kappa}_{3,3}(X)$ chosen in Example 2.10. It is with respect to this choice that we now define the subgroups $\boldsymbol{\kappa}^{\pm} \hookrightarrow \boldsymbol{\kappa}$ and the corresponding quotient maps $p^{\pm}$. The main theorem of this section is a refinement to the similarity 2nd Hasse-Witt invariant of the fact that the classical 2nd Hasse-Witt invariant of a reduced pfaffian is the class in the Brauer group of the corresponding Azumaya algebra.

**Theorem 3.5.** *Let $X$ be a noetherian scheme with $\frac{1}{2} \in \mathscr{O}_X$ considered in the étale topology. Let $(\mathscr{A}, \mathscr{P}, \varphi)$ be a 2-torsion datum of degree 4 on $X$. Then*

$$p_*^+ gw_2^{\mathscr{H}}(\mathscr{S}_\varphi^-(P), \mathrm{pf}_A, \mathrm{pf}(P), \zeta_{\mathscr{P}}) = a(\mathscr{A}, \mathscr{P}, \varphi),$$

*in $H^2_{\text{ét}}(X, \boldsymbol{\mu}_2)$, i.e. the similarity 2nd Hasse-Witt invariant of a reduced pfaffian is uniquely determined in $H^1_{\text{ét}}(X, \boldsymbol{\kappa})$ by the classes $a(\mathscr{A}, \mathscr{P}, \varphi)$ and $c_1(\mathrm{pf}(\mathscr{P}), \boldsymbol{\mu}_2)$ in $H^1_{\text{ét}}(X, \boldsymbol{\mu}_2)$.*

The proof links the reduced pfaffian and the similarity Hasse-Witt class via their interpretations of maps on étale cohomology induced from diagrams of group schemes. From the work we've already done, it's quite a simple diagram chase, combining the $n = 4$ version of (3.2), Theorem 3.3, and the key diagram given in the following proposition, which employs the construction of a particular half-spin representation of the Clifford group.

**Proposition 3.6.** *There exists a "half-spin" representation*

$$\rho^+ : \mathbf{S\Gamma}(\mathscr{A}lt_4) \to \mathbf{GL}_4,$$

*rendering commutative the following diagram of group schemes,*

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
1 \arrow[r] & \boldsymbol{\kappa}^+(\mathscr{A}lt_4) \arrow[r] \arrow[d] & \boldsymbol{\kappa}^+(\mathscr{A}lt_4) \arrow[d] & & \\
1 \arrow[r] & \boldsymbol{\kappa}(\mathscr{A}lt_4) \arrow[r] \arrow[d, "p^+"] & \mathbf{S\Gamma}(\mathscr{A}lt_4) \arrow[r, "s"] \arrow[d, "\rho^+"] & \mathbf{GSO}(\mathscr{A}lt_4) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \boldsymbol{\mu}_2 \arrow[r] \arrow[d] & \mathbf{GL}_4 \arrow[r, "\Phi"] \arrow[d] & \mathbf{GSO}(\mathscr{A}lt_4) \arrow[r] & 1 \\
 & 1 & 1 & &
\end{tikzcd}$$

*in the étale topology on $X$ with exact rows and columns.*

*Proof.* For the Clifford algebra of a split pfaffian form there's a canonical $\mathscr{O}_X$-algebra isomorphism defined on sections $x$ of $\mathscr{A}lt_4 \hookrightarrow \mathscr{C}(\mathscr{A}lt_4, \mathrm{pf})$ over $U \to X$, by

$$\begin{aligned}
\Psi \ : \ \mathscr{C}(\mathscr{A}lt_4(\mathscr{O}_X), \mathrm{pf}) \quad &\xrightarrow{\sim} \quad \mathscr{M}_2(\mathscr{M}_4(\mathscr{O}_X)) \\
x \quad &\longmapsto \quad \begin{pmatrix} 0 & \pi_0(x) \\ x & 0 \end{pmatrix},
\end{aligned}$$

where $\pi_0 : \mathscr{A}lt_4(\mathscr{O}_X) \to \mathscr{A}lt_4(\mathscr{O}_X)$ is the (amazing) $\mathscr{O}_X$-module morphism defined by

$$\pi_0 \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -a_{34} & a_{24} & -a_{23} \\ a_{34} & 0 & -a_{14} & a_{13} \\ -a_{24} & a_{14} & 0 & -a_{12} \\ a_{23} & -a_{13} & a_{12} & 0 \end{pmatrix}$$

and satisfying $x\pi_0(x) = \pi_0(x)x = \mathrm{pf}(x) \cdot \mathrm{id}$ for sections $x$ of $\mathscr{A}lt_4(\mathscr{O}_X)$ over $U \to X$. For all this, see Knus [34, V §5.2.1]. The $\mathscr{O}_X$-algebra isomorphism $\Psi$ induces the following canonical identifications: the even Clifford algebra (resp. and multiplicative group) with the block diagonal $\mathscr{O}_X$-subalgebra (resp. general linear group) of the matrix algebra,

$$\begin{aligned} \mathscr{C}_0(\mathscr{A}lt_4(\mathscr{O}_X), \mathrm{pf}) &\xrightarrow{\sim} \mathscr{M}_4(\mathscr{O}_X) \times \mathscr{M}_4(\mathscr{O}_X) \\ \mathscr{C}_0(\mathscr{A}lt_4(\mathscr{O}_X), \mathrm{pf})^\times &\xrightarrow{\sim} \mathbf{GL}_4 \times \mathbf{GL}_4 \end{aligned}$$

the even Clifford and spin groups with certain subgroups of invertible matrices,

$$\begin{aligned} \mathbf{S\Gamma}(\mathscr{A}lt_4(\mathscr{O}_X), \mathrm{pf}) &\xrightarrow{\sim} \left\{ (x, y) \in \mathbf{GL}_4 \times \mathbf{GL}_4 : \begin{array}{c} xy^t = a \cdot \mathrm{id}_4 \\ \det x = \det y = a^2 \end{array} \quad a \in \mathbb{G}_\mathrm{m} \right\} \\ \mathbf{Spin}(\mathscr{A}lt_4(\mathscr{O}_X), \mathrm{pf}) &\xrightarrow{\sim} \{ (x, (x^t)^{-1}) : x \in \mathbf{SL}_4 \} \end{aligned}$$

the Clifford norm sequence and vector representation sequence,

$$\begin{array}{rcl} 1 \to \mathbf{Spin}(\mathscr{A}lt_4(\mathscr{O}_X), \mathrm{pf}) \to \mathbf{S\Gamma}(\mathscr{A}lt_4(\mathscr{O}_X), \mathrm{pf}) & \xrightarrow{N} & \mathbb{G}_\mathrm{m} \to 1 \\ (x, y) & \mapsto & xy^t \end{array}$$

$$\begin{array}{rcl} 1 \to \mathbb{G}_\mathrm{m} \to \mathbf{S\Gamma}(\mathscr{A}lt_4(\mathscr{O}_X), \mathrm{pf}) & \xrightarrow{r} & \mathbf{SO}(\mathscr{A}lt_4(\mathscr{O}_X), \mathrm{pf}) \to 1 \\ (x, y) & \mapsto & z \mapsto yzx^{-1} \end{array}$$

and finally, the Clifford sequence for the orthogonal similitude group,

$$\begin{array}{rcl} 1 \to \boldsymbol{\kappa}(\mathscr{A}lt_4(\mathscr{O}_X), \mathrm{pf}) \to \mathbf{S\Gamma}(\mathscr{A}lt_4(\mathscr{O}_X), \mathrm{pf}) & \xrightarrow{s} & \mathbf{GSO}(\mathscr{A}lt_4(\mathscr{O}_X), \mathrm{pf}) \to 1 \\ (x, y) & \mapsto & z \mapsto yzy^t. \end{array}$$

Furthermore, there's a diagram of double covers of groups schemes,

$$\begin{tikzcd} & \mathbf{S\Gamma}(\mathscr{A}lt_4) \arrow[dl, "\rho^-"'] \arrow[d, "N \times r"] \arrow[dr, "\rho^\times"] & \\ \mathbf{GL}_4 \arrow[dr, "\det \cdot \Phi \circ (-)^{t-1}"'] & \mathbb{G}_\mathrm{m} \times \mathbf{SO}(\mathscr{A}lt_4) \arrow[d, "m"] & \mathbf{GL}_4 \arrow[dl, "\Phi"] \\ & \mathbf{GSO}(\mathscr{A}lt_4) & \end{tikzcd}$$

in the étale topology on $X$, where $\rho^-(x,y) = x$, $\rho^+(x,y) = y$, and $\Phi$ is the homomorphism

$$\begin{array}{rcl} \mathbf{GL}_4 & \to & \mathbf{GSO}(\mathscr{A}lt_4(\mathscr{O}_X), \mathrm{pf}) \\ y & \mapsto & z \mapsto yzy^t \end{array}$$

and where $\det \cdot \Phi \circ (-)^{t-1}$ is interpreted as

$$\begin{array}{rcl} \mathbf{GL}_4 & \to & \mathbf{GSO}(\mathscr{A}lt_4(\mathscr{O}_X), \mathrm{pf}) \\ x & \mapsto & z \mapsto \det(x)\,(x^t)^{-1} z x^{-1}. \end{array}$$

The kernels of $m$ and $\rho^\pm$ are canonically identified with the subgroups $\boldsymbol{\kappa}^0$ and $\boldsymbol{\kappa}^\pm$, respectively, where $\boldsymbol{\kappa} = \boldsymbol{\kappa}(\mathscr{A}lt_4(\mathscr{O}_X), \mathrm{pf})$ itself is canonically identified with $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 = \{(\pm\mathrm{id}, \pm\mathrm{id})\} \hookrightarrow \mathbf{S\Gamma}(\mathscr{A}lt_4(\mathscr{O}_X), \mathrm{pf})$. A rearrangement of this diagram – using the above canonical identification – focusing on $\rho^+$ yields the proposition. $\square$

*Remark* 3.7. The case of arbitrary Arf invariant will be treated in future work in the spirit of Knus, Parimala, and Sridharan [32].

# Chapter 4

# Future directions and open questions

## 4.1 Further investigation of the similarity Hasse-Witt invariant

There is still much work ahead to fully understand the general classification of $\mathscr{L}$-valued symmetric bilinear forms, properties of the invariant $gw_2$, and theory of other cohomological invariants of these forms.

*a)* Describe a "splitting principle" for line bundle-valued symmetric bilinear forms. If $\mathscr{L}$ is not a square in the Picard group, $\mathscr{L}$-valued symmetric bilinear spaces are necessarily of even rank. Given an $\mathscr{L}$-valued form, can one always find a morphism $f : Y \to X$ that is injective on cohomology and such that the pull back form is a sum of hyperbolic planes? Use this to give a nice characterization of how the invariant $gw_2$ behaves with respect to Whitney sums.

*b)* Since our invariant is only defined for forms of a fixed discriminant, we've taken great care to keep track of the the base form. When $\mathscr{L}$ is not a square, there's a question as to what values in $H^1_{\text{ét}}(X, \boldsymbol{\mu}_2)$ can the discriminant of $\mathscr{L}$-valued forms take. One possible guess (at least for the case of varieties over a field) is that all $\mathscr{L}$-valued forms have trivial Arf invariant, i.e. discriminant isometry to a discriminant of a hyperbolic form. Note that this guess would indeed follow from an affirmative answer to the question in *a)* concerning splitting in terms of hyperbolic planes. We ask for examples of $\mathscr{L}$-valued forms with $\mathscr{L}$ not a square and with nontrivial Arf invariant.

*c)* Complete the investigation of $\mathscr{L}$-valued symmetric bilinear forms of rank 6 with arbitrary Arf invariant.

*d)* Relating the invariant $gw_2$ of $\mathscr{L}$-valued symmetric bilinear forms of rank 4 to the corresponding classes of quaternion algebras, following Knus, Parimala, and Sridharan [32] and Bichsel and Knus [7].

*e)* How is the invariant $gw_2^{\mathscr{H}}(\mathscr{E}, q, \mathscr{L})$ related to the class of the generalized even Clifford algebra $\mathscr{C}_0(\mathscr{E}, b, \mathscr{L})$, in the corresponding hermitian involutive Brauer group of Parimala and Srinivas [41]?

*f)* Computing the invariant $gw_2$ for other arithmetically and geometrically significant $\mathscr{L}$-valued forms not occurring in the above considered general families, e.g. those arising from octonian algebras and the cohomology of varieties.

*g)* Providing a "universal" description of the invariant $gw_2$ in the spirit of Jardine [27]. This would require a calculation of $H^2_{\text{ét}}(B\mathbf{GSO}, \boldsymbol{\kappa})$. The mod 2 cohomology of $B\mathbf{SO}$ is a polynomial ring in the universal Hasse-Witt invariants. The multiplier sequence induces a fibration $\mathbb{G}_{\mathrm{m}} \to B\mathbf{SO} \to B\mathbf{GSO}$ on the level of simplicial classifying schemes. Holla and Nitsure [25], [26] compute the action of $\mathbb{G}_{\mathrm{m}}$ on the universal Hasse-Witt classes to give a presentation of the mod 2 cohomology ring of $B\mathbf{GO}$ in terms of generators and relations. Similar methods will work to calculate the cohomology of $B\mathbf{GSO}$ with coefficients in $\boldsymbol{\kappa}$ (and it's tensor powers). This first requires explicit generators of the "mod 4" cohomology of $B\mathbf{SO}$. In particular, we would ask for a calculation of $H^2_{\text{ét}}(B\mathbf{GSO}_{m,m}, \boldsymbol{\mu}_4)$ for $m$ even, and in general, $H^i_{\text{ét}}(B\mathbf{GSO}_{m,m}, \boldsymbol{\mu}_4^{\otimes i})$ to perhaps find higher invariants for line bundle-valued forms.

## 4.2 Isometry class cohomological invariants

We ask for the precise relationship of the 2nd coboundary of the outer twisted spin cover (see Theorem 1.40)
$$1 \to \boldsymbol{\mu}_2 \to \mathbf{Spin}(H_{\mathscr{L}}(\mathscr{O}_X^m)) \to \mathbf{SO}(H_{\mathscr{L}}(\mathscr{O}_X^m)) \to 1$$
in terms of the invariants already described.

Fix a line bundle $\mathscr{L}$ on a scheme $X$. The following holds for general $n$ even, but later we will care only about the case when $\mathscr{L}$ is not a square in the Picard group. For each element $\delta \in H^1_{\text{ét}}(X, \boldsymbol{\mu}_2)$ such that there exists an $\mathscr{L}$-valued symmetric bilinear space of even rank $n$ and with discriminant $\delta$, fix such a form $\mathscr{T}_{\mathscr{L}} = (\mathscr{T}_{\mathscr{L}}, t_{\mathscr{L}}, \mathscr{L})$. Let $(\mathscr{E}, b, \mathscr{L}, \zeta)$ be an oriented $\mathscr{L}$-valued symmetric bilinear space of rank $n$ with orientation $\zeta : \operatorname{disc}(\mathscr{T}_{\mathscr{L}}) \xrightarrow{\sim} \operatorname{disc}(\mathscr{E}, b, \mathscr{L})$. Let $(\mathscr{H}, h)$ be any $\mathscr{O}_X$-valued symmetric bilinear form of rank $n$ with discriminant $\delta$ (we can always find such an $\mathscr{O}_X$-valued form) and choose orientations $\zeta_{\mathscr{E}} : \operatorname{disc}(\mathscr{H}, h) \xrightarrow{\sim} \operatorname{disc}(\mathscr{E}, b, \mathscr{L})$ and $\zeta_{\mathscr{L}} : \operatorname{disc}(\mathscr{H}, h) \to \operatorname{disc}(\mathscr{T}_{\mathscr{L}})$ such that $\zeta_{\mathscr{E}} = \zeta \circ \zeta_{\mathscr{L}}$. We will show that the element,
$$gw_2^{\mathscr{H}}(\mathscr{E}, b, \mathscr{L}, \zeta_{\mathscr{E}}) - gw_2^{\mathscr{H}}(\mathscr{T}_{\mathscr{L}}, \zeta_{\mathscr{L}}),$$
in the image of $H^2_{\text{ét}}(X, \boldsymbol{\mu}_2) \to H^2_{\text{ét}}(X, \boldsymbol{\kappa}(\mathscr{H}, h))$, is independent of $(\mathscr{H}, h)$ and the orientations $\zeta_{\mathscr{E}}$ and $\zeta_{\mathscr{L}}$, and so only depends on $\mathscr{T}_{\mathscr{L}}$ and the orientation $\zeta$. Indeed, for any other choice of $(\mathscr{H}', h')$ of rank $n$ and discriminant $\delta$ and orientations $\zeta'_{\mathscr{E}} : \operatorname{disc}(\mathscr{H}', h') \xrightarrow{\sim} \operatorname{disc}(\mathscr{E}, b, \mathscr{L})$ and $\zeta'_{\mathscr{L}} : \operatorname{disc}(\mathscr{H}', h') \to \operatorname{disc}(\mathscr{T}_{\mathscr{L}})$ such that $\zeta'_{\mathscr{E}} = \zeta \circ \zeta'_{\mathscr{L}}$, by the comparison formula of Theorem 2.18, we have that
$$gw_2^{\mathscr{H}'}(\mathscr{E}, b, \mathscr{L}, \zeta'_{\mathscr{E}}) = \iota_*^{\mathscr{H}'} w_2^{\mathscr{H}'}(\mathscr{H}, h) + ({\zeta'_{\mathscr{E}}}^{-1} \circ \zeta_{\mathscr{E}})_* gw_2^{\mathscr{H}}(\mathscr{E}, b, \mathscr{L}, \zeta_{\mathscr{E}})$$
and
$$gw_2^{\mathscr{H}'}(\mathscr{T}_{\mathscr{L}}, \zeta'_{\mathscr{L}}) = \iota_*^{\mathscr{H}'} w_2^{\mathscr{H}'}(\mathscr{H}, h) + ({\zeta'_{\mathscr{L}}}^{-1} \circ \zeta_{\mathscr{L}})_* gw_2^{\mathscr{H}}(\mathscr{T}_{\mathscr{L}}, \zeta_{\mathscr{L}})$$
in $H^2_{\text{ét}}(X, \boldsymbol{\kappa}(\mathscr{H}', h'))$. Noting that ${\zeta'_{\mathscr{E}}}^{-1} \circ \zeta_{\mathscr{E}} = {\zeta'_{\mathscr{L}}}^{-1} \circ \zeta_{\mathscr{L}}$, we have that
$$gw_2^{\mathscr{H}'}(\mathscr{E}, b, \mathscr{L}, \zeta'_{\mathscr{E}}) - gw_2^{\mathscr{H}'}(\mathscr{T}_{\mathscr{L}}, \zeta'_{\mathscr{L}}) = ({\zeta'_{\mathscr{L}}}^{-1} \circ \zeta_{\mathscr{L}})_* \big(gw_2^{\mathscr{H}}(\mathscr{E}, b, \mathscr{L}, \zeta_{\mathscr{E}}) - gw_2^{\mathscr{H}}(\mathscr{T}_{\mathscr{L}}, \zeta_{\mathscr{L}})\big)$$

inside the image of $H^2_{\text{ét}}(X, \boldsymbol{\mu}_2) \to H^2_{\text{ét}}(X, \boldsymbol{\kappa}(\mathscr{H}', h'))$. We are looking for a common canonical lifting of this element to $H^2_{\text{ét}}(X, \boldsymbol{\mu}_2)$.

Our question is how does such a canonical lift relate the 2nd coboundary map,

$$\begin{array}{rcl} w_2^{\mathscr{T}_{\mathscr{L}}} : H^1_{\text{ét}}(X, \mathbf{SO}(\mathscr{T}_{\mathscr{L}})) & \to & H^2_{\text{ét}}(X, \boldsymbol{\mu}_2) \\ [\mathscr{E}, b, \mathscr{L}, \zeta] & \mapsto & w_2^{\mathscr{T}_{\mathscr{L}}}(\mathscr{E}, b, \mathscr{L}, \zeta) \end{array}$$

of the (outer) twisted spin cover,

$$1 \to \boldsymbol{\mu}_2 \to \mathbf{Spin}(\mathscr{T}_{\mathscr{L}}) \to \mathbf{SO}(\mathscr{T}_{\mathscr{L}}) \to 1$$

in the étale topology on $X$, defined in §1.5.2 Theorem 1.40?

## 4.3 Transfers in Grothendieck-Witt theory

There has been much recent activity in the calculation of Grothendieck-Witt groups of schemes. The proper transfer homomorphism has played a strong role in these calculations. For example, Walter [51] following work of Arason [1], Szyjewski [49], and Gill [21], calculates all derived Grothendieck-Witt groups of projective bundles. As for other projective homogeneous varieties, Pumplün [42], [43] handles the case of Brauer-Severi varieties, while Balmer and Calmès [6] handle Grassmann varieties.

As Grothendieck-Witt groups of schemes are explicitly calculated, the dependence of the invariant $gw_2$ on the relevant geometry can be more fully investigated. Also important is the behavior of $gw_2$ under the transfer homomorphism. This is a generalization of the study of invariants of trace forms in number theory on the one hand, and of the study of the invariants of intersection pairings on varieties of dimension divisible by 4 on the other. Understanding this behavior will give insight into the "orthogonal Riemann-Roch" question posed by Shapiro [48] and by Taylor [50]. Namely, given a proper morphism $f : X \to Y$ of schemes of relative dimension $d$ divisible by 4, does there exist a diagram of the form,

$$\begin{tikzcd} GW^d(X, \omega_f) \arrow[r, "\text{``}gw\text{''}"] \arrow[d, "f_*"] & H^*_{\text{ét}}(X) \arrow[d, "\text{``}f_*\text{''}"] \\ GW^0(Y, \mathscr{O}_Y) \arrow[r, "w"] & H^*_{\text{ét}}(Y, \boldsymbol{\mu}_2) \end{tikzcd}$$

for some theory of cohomology $H^*_{\text{ét}}(X)$, theory of cohomological invariants "$gw$" (perhaps with some defect factor included), and some theory of cohomological transfer "$f_*$", only depending on the morphism $f$? The theory of $gw_2$ is a first approximation at filling in the top portion of this kind of diagram. Strictly speaking, $gw_2$ does not naively descend to the Grothendieck-Witt group (due to the dependence on $\boldsymbol{\kappa}$) but it might descend to a certain refinement of the Grothendieck-Witt group, e.g. by keeping track of orientation. Perhaps a lesson is that it's necessary to expand the coefficients taken in étale cohomology beyond $\boldsymbol{\mu}_2$, in order to define cohomological invariants of $\omega_f$-valued forms.

# Appendix A

# The categorical language of torsors

## A.1 Torsors

Let $X$ be a scheme with a fixed (Grothendieck) topology. By a sheaf on $X$ we mean a sheaf of sets on the site associated to this topology. Let $\mathbf{G}$ be a sheaf of groups on $X$ with a fixed structural morphism $\mathbf{G} \to X$, considering $X$ as a sheaf on $X$ via its functor of points. A *(right)* $\mathbf{G}$*-sheaf* on $X$ is a triple $(\mathscr{E}, \cdot, p)$, where $\mathscr{E}$ is a sheaf on $X$ with a (right) $\mathbf{G}$-action $\mathscr{E} \times_X \mathbf{G} \xrightarrow{\cdot} \mathscr{E}$ and a $\mathbf{G}$-invariant morphism of sheaves $p : \mathscr{E} \to X$, called the *projection*. By abuse of notation we will often write $\mathscr{E}$ in place of $(\mathscr{E}, \cdot, p)$. Morphisms of $\mathbf{G}$-sheaves are $\mathbf{G}$-equivariant functors. The *trivial* $\mathbf{G}$-sheaf is $\mathbf{G}$ itself, together with its own (right) multiplication action and fixed structural morphism as projection.

**Definition A.1.** A (right) $\mathbf{G}$-sheaf $\mathscr{E}$ is a *(right) sheaf* $\mathbf{G}$*-torsor* over $X$ if there exists a cover $\mathscr{U} = \{U_i \to X\}_{i \in I}$ of $X$ so that $\mathscr{E}$ is trivial over $\mathscr{U}$, i.e. for each $i \in I$, the $\mathbf{G}|_{U_i}$-sheaf $\mathscr{E}|_{U_i}$ is isomorphic to the trivial $\mathbf{G}|_{U_i}$-sheaf on $U_i$.

Note that the notion of sheaf torsor depends on the topology on $X$. The category $\mathsf{Tors}(\mathbf{G})$, of sheaf torsors for $\mathbf{G}$ over $X$ is a full subcategory of the category of $\mathbf{G}$-sheaves and is a groupoid, see Demazure/Gabriel [14, III §4.1.3].

**Proposition A.2.** *Let $X$ be a scheme, $\mathbf{G}$ a sheaf of groups on $X$, and $f : \mathscr{E} \to \mathscr{E}'$ a morphism of $\mathbf{G}$-sheaves on $X$. If $\mathscr{E}$ and $\mathscr{E}'$ are sheaf torsors for $\mathbf{G}$ over $X$ then $f$ is an isomorphism.*

Some well known properties of sheaf torsors are corollaries of this fact, again see Demazure/ Gabriel [14, III §4.1.5-7].

**Corollary A.3.** *Let $X$ be a scheme, $\mathbf{G}$ a sheaf of groups on $X$, and $\mathscr{E}$ a $\mathbf{G}$-sheaf on $X$.*

*a) Then $\mathscr{E}$ is trivial if and only if $\mathscr{E}$ is a sheaf torsor for $\mathbf{G}$ over $X$ and the projection $p : \mathscr{E} \to X$ has a section.*

*b) Then $\mathscr{E}$ is a sheaf $\mathbf{G}$-torsor over $X$ if and only if the projection $p : \mathscr{E} \to X$ is an epimorphism of sheaves on $X$ and the canonical morphism of sheaves on $X$,*

$$\mathscr{E} \times_X \mathbf{G} \to \mathscr{E} \times_X \mathscr{E},$$

*defined on sections over $U \to X$ by $(y, g) \mapsto (y, y \cdot g)$, is an isomorphism of sheaves on $X$.*

If $\mathscr{E}$ is a sheaf $\mathbf{G}$-torsor over $X$, then we call $\mathscr{E}$ a *principal homogeneous space* for $\mathbf{G}$ over $X$ (or $\mathbf{G}$*-torsor* over $X$) if $\mathscr{E}$ is representable by a scheme over $X$. It is a subtle matter to decide which sheaf torsors are representable, for instance see Milne [36, III, Theorem 4.3, Remark 4.4].

**Theorem A.4.** *Let $X$ be a scheme, $\mathbf{G}$ a group scheme over $X$.*

*a) If $X$ has the flat topology and $\mathbf{G}$ is flat, locally of finite-type, and affine over $X$, then every sheaf $\mathbf{G}$-torsor over $X$ is representable by a scheme.*

*b) If $X$ has the étale topology and $\mathbf{G}$ is smooth, locally of finite-type, and affine over $X$, then every sheaf $\mathbf{G}$-torsor over $X$ is representable by a scheme.*

## A.2 Twisted forms

Let $X$ be a scheme with a topology and $\mathbf{G}$ be a sheaf of groups on $X$. For the following general theorem, see Giraud [23, Théorème 1.4.5, Exemple 2.1.2, Corollaire 2.2.6].

**Theorem A.5.** *Let $X$ be a scheme with a topology and $\mathbf{G}$ be a sheaf of group on $X$.*

*a) For any object $\mathscr{E}$ of $\mathsf{Tors}(\mathbf{G})$, the presheaf of groups $\mathbf{Aut}_{\mathbf{G}}(\mathscr{E})$ defined by*

$$\mathbf{Aut}_{\mathbf{G}}(\mathscr{E})(U) = \mathrm{Aut}_{\mathsf{Tors}(\mathbf{G})}(\mathscr{E}|_U)$$

*over $U \to X$, is a sheaf of groups on $X$.*

*b) For any objects $\mathscr{E}$ and $\mathscr{E}'$ of $\mathsf{Tors}(\mathbf{G})$, the presheaf of sets $\mathbf{Isom}_{\mathbf{G}}(\mathscr{E}, \mathscr{E}')$ defined by*

$$\mathbf{Isom}_{\mathbf{G}}(\mathscr{E}, \mathscr{E}')(U) = \mathrm{Isom}_{\mathsf{Tors}(\mathbf{G})}(\mathscr{E}|_U, \mathscr{E}'|_U)$$

*over $U \to X$, is a sheaf on $X$ and has a canonical structure of (right) sheaf torsor for $\mathbf{Aut}_{\mathbf{G}}(\mathscr{E})$ over $X$.*

*c) There's a canonical isomorphism,*

$$\mathbf{G} \to \mathbf{Aut}_{\mathbf{G}}(\mathbf{G}),$$

*of sheaves of groups on $X$, where on the right, $\mathbf{G}$ denotes the trivial sheaf torsor for $\mathbf{G}$ over $X$.*

More generally, for each $U \to X$, $\mathsf{Tors}(\mathbf{G}|_U)$ is a category over $U$. We denote by $\mathsf{TORS}(\mathbf{G})$ the associated fibered category over the topology on $X$. Then $\mathsf{TORS}(\mathbf{G})$ is a *stack*, see Giraud [23, II] for a precise definition, but colloquially we say that $\mathsf{TORS}(\mathbf{G})$ satisfies descent. Conversely, we fix a stack $\mathsf{T}$ over the topology on $X$ (e.g. the stack associated to the category of schemes over $X$, sheaves of groups on $X$, vector bundles on $X$, or bilinear forms on $X$, etc). Let $\mathscr{E}$ be an object of $T|_X$. We call an object $\mathscr{E}'$ of $T|_X$ a *(twisted) form* of $\mathscr{E}$ if these objects are locally isomorphic, i.e. if there exists a cover $\mathscr{U}$ of $X$ so that $\mathscr{E}$ and $\mathscr{E}'$ are isomorphic when restricted to $\mathscr{U}$. Denote by $\mathsf{Forms}(\mathscr{E})$ the category of forms of $\mathscr{E}$ and $\mathsf{FORMS}(\mathscr{E})$ the associated sub-stack of $\mathsf{T}$ of twisted forms of $\mathscr{E}$, i.e. $\mathsf{FORMS}(\mathscr{E})|_U = \mathsf{Forms}(\mathscr{E}|_U)$ for $U \to X$. For an object $\mathscr{E}$ in $\mathsf{T}|_X$ denote by $\mathrm{Forms}(\mathscr{E})$ be the set of isomorphism classes of forms of $\mathscr{E}$ over $X$, which is a pointed set with distinguished element the class of $\mathscr{E}$.

### A.2.1 Twisted forms vs. torsors

In parallel with Theorem A.5, we can compare twisted forms and torsors for the sheaf of automorphism group, see Giraud [23, Théorème 2.5.1].

**Theorem A.6.** *Let $\mathsf{T}$ be a stack over the topology of a scheme $X$ and let $\mathscr{E}$ be an object of $\mathsf{T}|_X$. Let $\mathbf{Aut}(\mathscr{E})$ be the sheaf on $X$ of automorphism groups of $\mathscr{E}$ in $T$. Then there's an equivalence of stacks $\mathsf{FORMS}(\mathscr{E}) \to \mathsf{TORS}(\mathbf{Aut}(\mathscr{E}))$, given by the equivalences of categories*

$$\begin{array}{rcl} \mathsf{Forms}(\mathscr{E}|_U) & \to & \mathsf{Tors}(\mathbf{Aut}(\mathscr{E}|_U)) \\ \mathscr{E}' & \mapsto & \mathbf{Isom}_{\mathbf{Aut}(\mathscr{E}|_U)}(\mathscr{E}|_U, \mathscr{E}') \end{array}$$

*for $U \to X$. In particular, there's an equivalence of categories $\mathsf{Forms}(\mathscr{E}) \to \mathsf{Tors}(\mathbf{Aut}(\mathscr{E}))$.*

### A.2.2 The case of symmetric bilinear forms

We consider the case of symmetric bilinear forms. The case of general bilinear forms can be treated similarly.

**Theorem A.7.** *Let $X$ be a scheme with $\frac{1}{2} \in \mathscr{O}_X$ considered in the étale topology. Let $(\mathscr{H}, h, \mathscr{L})$ be a fixed $\mathscr{L}$-valued symmetric bilinear space of rank $n$ on $X$.*

*a) The category of $\mathbf{O}(\mathscr{E}, b, \mathscr{L})$-torsors is equivalent to the category of whose objects are $\mathscr{L}$-valued symmetric bilinear spaces of rank $n$ and whose morphisms are isometries.*

*b) The category of $\mathbf{SO}(\mathscr{E}, b, \mathscr{L})$-torsors is equivalent to the category whose objects are pairs $((\mathscr{E}', b', \mathscr{L}), \psi')$ consisting of an $\mathscr{L}$-valued symmetric bilinear space of rank $n$ together with an isometry $\psi' : \mathrm{disc}(\mathscr{E}, b, \mathscr{L}) \to \mathrm{disc}(\mathscr{E}', b', \mathscr{L})$ of discriminant forms, and whose morphisms between objects $((\mathscr{E}', b', \mathscr{L}), \psi')$ and $((\mathscr{E}'', b'', \mathscr{L}), \psi'')$ are isometries $\varphi : (\mathscr{E}', b', \mathscr{L}) \to (\mathscr{E}'', b'', \mathscr{L})$ such that $\psi'' = \mathrm{disc}(\varphi) \circ \psi'$.*

*c) The category of $\mathbf{GO}(\mathscr{E}, b, \mathscr{L})$-torsors is equivalent to the category whose objects are all symmetric bilinear spaces of rank $n$ with values in a line bundle and whose morphisms are similarity transformations.*

*d) Let $n$ be even. The category of $\mathbf{GSO}(\mathscr{E}, b, \mathscr{L})$-torsors is equivalent to the category whose objects are pairs $((\mathscr{E}', b', \mathscr{L}'), \psi')$ consisting of an $\mathscr{L}'$-valued symmetric bilinear space of rank $n$ (for some line bundle $\mathscr{L}'$ on $X$) together with an isometry $\psi' : \mathrm{disc}(\mathscr{E}, b, \mathscr{L}) \to \mathrm{disc}(\mathscr{E}', b', \mathscr{L}')$ of discriminant forms, and whose morphisms between any two objects $((\mathscr{E}', b', \mathscr{L}'), \psi')$ and $((\mathscr{E}'', b'', \mathscr{L}''), \psi'')$ are similarity transformations $\varphi : (\mathscr{E}', b', \mathscr{L}') \to (\mathscr{E}'', b'', \mathscr{L}'')$ such that $\psi'' = \mathrm{disc}(\varphi) \circ \psi'$.*

*Moreover, the above sheaves of groups are smooth, locally of finite-type, and affine over $X$, hence all respective sheaf torsors are representable as schemes.*

*Proof.* In each case, we identify the stated category of sheaf torsors with a corresponding category of twisted forms (inside the sub-stack of $\mathscr{O}_X$-modules that locally have the structure of a bilinear form) of a base object, then we'll appeal to Theorem A.6. There are two steps. First, we show that the sheaf of automorphism groups of the base object is isomorphic to the stated sheaf of groups.

Second, we identify all twisted forms of the corresponding base object. To this end, we prove that every object of the stated form is a twisted form of the base object by finding a suitable covering, we then prove that every twisted form is of the stated type by using properties of the contracted product.

For *a)* and *c)*, first note that $\mathbf{O}(\mathscr{H}, h, \mathscr{L})$ (resp. $\mathbf{GO}(\mathscr{H}, h, \mathscr{L})$) is defined as the sheaf of isometry (resp. similitude) groups of $(\mathscr{H}, h, \mathscr{L})$. Second, let $(\mathscr{E}, b, \mathscr{M})$ be a symmetric bilinear space of rank $n$ on $X$ and let $\mathscr{U} = \{U_i \to X\}_{i\in I}$ be an étale cover of $X$ trivializing both $\mathscr{L}$ and $\mathscr{M}$ via $l_i : \mathscr{O}_{U_i} \xrightarrow{\sim} \mathscr{L}|_{U_i}$ and $m_i : \mathscr{O}_{U_i} \xrightarrow{\sim} \mathscr{M}|_{U_i}$. Since every $\mathscr{O}_X$-valued symmetric bilinear form is locally isomorphic for the étale topology, see for example Demazure/Gabriel [14, III, §5.2], we can refine the cover $\mathscr{U}$, finding isometries $\varphi_i : (\mathscr{H}|_{U_i}, l_i^{-1} \circ h|_{U_i}, \mathscr{O}_{U_i}) \xrightarrow{\sim} (\mathscr{E}|_{U_i}, m_i^{-1} \circ b|_{U_i}, \mathscr{O}_{U_i})$ for each $i \in I$. Now note that these isometries induce similarities $(\varphi_i, m_i \circ l_i^{-1}) : (\mathscr{H}|_{U_i}, h|_{U_i}, \mathscr{L}|_{U_i}) \xrightarrow{\sim} (\mathscr{E}|_{U_i}, b|_{U_i}, \mathscr{M}|_{U_i})$, which are themselves isometries of the corresponding line bundle-valued forms if and only if $\mathscr{M}|_{U_i} = \mathscr{L}|_{U_i}$ and $m_i \circ l_i^{-1}$ is the identity map for all $i \in I$, i.e. $\mathscr{M} = \mathscr{L}$. Thus $(\mathscr{H}, h, \mathscr{L})$ and $(\mathscr{E}, b, \mathscr{M})$ are locally similar in the étale topology, and are locally isometric if and only if $\mathscr{M} = \mathscr{L}$. Now we prove that every twisted form $\mathscr{E}$ (i.e. $\mathscr{O}_X$-module with the structure of a bilinear form on some étale cover) of $(\mathscr{H}, h, \mathscr{L})$ has the structure of a symmetric bilinear form with values in $\mathscr{L}$ (resp. a line bundle) on $X$. To this end, let $\mathbf{G}$ be $\mathbf{O}(\mathscr{H}, h, \mathscr{L})$ (resp. $\mathbf{GO}(\mathscr{H}, h, \mathscr{L})$) and consider the corresponding (sheaf) $\mathbf{G}$-torsor $P = \mathbf{Isom}_{\mathbf{G}}(\mathscr{E}, (\mathscr{H}, h, \mathscr{L}))$ of isometries (resp. similitudes). Then there's a canonical isomorphism of $\mathscr{O}_X$-modules $P \overset{\mathbf{G}}{\wedge} \mathscr{H} \xrightarrow{\sim} \mathscr{E}$. The map $P \times (\mathscr{H} \otimes \mathscr{H}) \xrightarrow{\mathrm{id}\times h} P \times \mathscr{L}$ induces a symmetric $\mathscr{O}_X$-bilinear morphism,

$$P \overset{\mathbf{G}}{\wedge} (\mathscr{H} \otimes \mathscr{H}) \to P \overset{\mathbf{G}}{\wedge} \mathscr{L},$$

where $\mathscr{L}$ has the structure of left $\mathbf{G}$-sheaf via the multiplier coefficient. In particular, if $\mathbf{G} = \mathbf{O}(\mathscr{H}, h, \mathscr{L})$, then by definition $\mathbf{G}$ acts trivially on $\mathscr{L}$ and there's a canonical isomorphism $P \overset{\mathbf{G}}{\wedge} \mathscr{L} \xrightarrow{\sim} \mathscr{L}$, thus there's a symmetric $\mathscr{O}_X$-bilinear morphism $\mathscr{E} \otimes \mathscr{E} \to \mathscr{L}$. If $\mathbf{G} = \mathbf{GO}(\mathscr{H}, h, \mathscr{L})$, then there are canonical $\mathscr{O}_X$-module morphisms,

$$P \overset{\mathbf{G}}{\wedge} \mathscr{L} \xrightarrow{\sim} (P \overset{\mathbf{G}}{\wedge} \mathbb{G}_{\mathrm{m}}) \overset{\mathbb{G}_{\mathrm{m}}}{\wedge} \mathscr{L} = {}^{\mu}P \overset{\mathbb{G}_{\mathrm{m}}}{\wedge} \mathscr{L},$$

where ${}^{\mu}P$ is the canonical (sheaf) $\mathbb{G}_{\mathrm{m}}$-torsor induced from $P$ by extension of structure group $\mathbf{GO}(\mathscr{H}, h, \mathscr{L}) \xrightarrow{\mu} \mathbb{G}_{\mathrm{m}}$. In particular, $P \overset{\mathbf{G}}{\wedge} \mathscr{L}$ is some line bundle $\mathscr{M}$, and thus there's a symmetric $\mathscr{O}_X$-bilinear morphism $\mathscr{E} \otimes \mathscr{E} \to \mathscr{M}$. □

## A.3 Nonabelian Čech cohomology

We'll first review the necessary notation of nonabelian cohomology in the spirit of Serre [47, §5], but valid for the étale site over $X$. We mostly follow Giraud [23, III §3].

**Definition A.8.** Let $\mathscr{U} = \{U_i \to X\}_{i\in I}$ be an étale cover of $X$ and $\mathbf{G}$ be a sheaf of groups on $X$.

- For each multi-index $(i_1, \dots, i_n) \in I^n$, let $U_{i_1\dots i_n} = U_{i_1} \times_X U_{i_2} \times_X \cdots \times_X U_{i_n}$.

- For each section $s_{i_1 \dots i_n} \in \mathbf{G}(U_{i_1 \dots i_n})$ and $1 \leq m \leq n$, let $s_{i_1 \dots i_{m-1} \underline{i_m} i_{m+1} \dots i_n}$ denote the image of $s_{i_1 \dots i_n}$ under $\mathbf{G}(U_{i_1 \dots i_{m-1} i_{m+1} \dots i_n}) \to \mathbf{G}(U_{i_1 \dots i_n})$ induced from the projection $U_{i_1 \dots i_n} \to U_{i_1 \dots i_{m-1} i_{m+1} \dots i_n}$.

- We'll call a family $(u_{ij})_{(i,j) \in I^2}$ of sections $u_{ij} \in \mathbf{G}(U_{ij})$ an étale Čech *1-cocycle* for $\mathscr{U}$ with valued in $\mathbf{G}$ if for each $(i,j,k) \in I^3$ we have the equality
$$u_{ij\underline{k}} \, u_{\underline{i}jk} = u_{i\underline{j}k},$$
in $\mathbf{G}(U_{ijk})$.

- We say that two Čech 1-cocycles $(u_{ij})$ and $(u'_{ij})$ for $\mathscr{U}$ with valued in $\mathbf{G}$ are *cohomologous* if there exists a family $(a_i)_{i \in I}$ of sections $a_i \in \mathbf{G}(U_i)$ such that for each $(i,j) \in I^2$ we have the equality,
$$u'_{ij} = a_{i\underline{j}} \, u_{ij} \, (a_{\underline{i}j})^{-1},$$
in $\mathbf{G}(U_{ij})$. This is an equivalence relation on the set of all 1-cocycles for $\mathscr{U}$.

- The set of all cohomology (equivalence) classes of Čech 1-cocycles for $\mathscr{U}$ with values in $\mathbf{G}$ is denoted $H^1_{\text{ét}}(\mathscr{U}/X, \mathbf{G})$, and is a pointed set with distinguished element the class of the constant identity 1-cocycle.

- Define the étale Čech cohomology set by the direct limit under refinement of covers,
$$H^1_{\text{ét}}(X, \mathbf{G}) = \varinjlim_{\mathscr{U}} H^1_{\text{ét}}(\mathscr{U}/X, \mathbf{G})$$
it's a pointed set with distinguished element the direct limit of the constant identity 1-cocycles.

### A.3.1 Torsors vs. Čech cohomology

Under certain conditions, torsors locally trivial in the étale topology and nonabelian étale Čech cohomology sets are in bijection, see for instance, Milne [36, Chapter III, Remark 4.8].

**Theorem A.9.** *Let* $\mathbf{G}$ *be a smooth affine group scheme over* $X$ *considered in the étale topology, then there's a canonical bijection,*
$$\mathrm{Tors}(\mathbf{G}) \to H^1_{\text{ét}}(X, \mathbf{G}),$$
*of pointed sets.*

*Proof.* In fact, it's true that for a given étale cover $\mathscr{U} = \{U_i \to X\}_{i \in I}$ of $X$, the set of isomorphism classes of torsors for $\mathbf{G}$ over $X$ that become trivial over $\mathscr{U}$ is in bijection with the cohomology set $\check{H}^1_{\text{ét}}(\mathscr{U}/X, \mathbf{G})$. A map is defined as follows. $\square$

### A.3.2 Twisted forms vs. Čech cohomology

Combining Theorems A.6 and A.9, we arrive at a direct comparison between isomorphism classes of twisted forms and Čech 1-cohomology classes.

**Theorem A.10.** *Let $\mathsf{T}$ be a stack over the étale topology on $X$ and let $\mathscr{E}$ be an object of $\mathsf{T}|_X$. Assume that $\mathbf{Aut}(\mathscr{E})$ is a smooth affine group scheme over $X$. Then there's a canonical bijection*

$$\mathrm{Forms}(\mathscr{E}) \to \check{H}^1_{\text{ét}}(X, \mathbf{Aut}(\mathscr{E}))$$

*of pointed sets.*

### A.3.3 Exact sequences in nonabelian cohomology

**Definition A.11.** Let $(A, e_A)$, $(B, e_B)$, and $(C, e_C)$ be pointed sets and $A \xrightarrow{f} B \xrightarrow{g} C$ a sequence of morphisms of pointed sets, i.e. set maps with $f(e_A) = e_B$ and $g(e_B) = e_C$. Then the sequence is called *exact* if $\ker(g) = \mathrm{im}(f)$, where $\ker(g) = g^{-1}(e_C)$. The sequence is called *strongly exact* if it is exact and furthermore, there's an action of $A$ on $B$ – a map $A \times B \xrightarrow{\cdot} B$ with $e_A \cdot b = b$ for all $b \in B$ – so that the orbits of $A$ in $B$ are the fibers of $g$, i.e. for each $c \in C$, and $b \in g^{-1}(c)$, $A \cdot b = g^{-1}(c)$.

**Proposition A.12.** *Let $1 \to \mathbf{G}' \to \mathbf{G} \to \mathbf{G}'' \to 1$ be an exact sequence of sheaves of groups on $X$. Then there's an associated strongly exact sequence of pointed sets,*

$$1 \to \mathbf{G}'(X) \to \mathbf{G}(X) \to \mathbf{G}''(X) \xrightarrow{d^0} H^1_{\text{ét}}(X, \mathbf{G}') \to H^1_{\text{ét}}(X, \mathbf{G}) \to H^1_{\text{ét}}(X, \mathbf{G}'').$$

*Furthermore, if $\mathbf{G}' \to \mathbf{G}$ is central, then the the above exact sequence of sets has a strongly exact extension to the right,*

$$\cdots \to \check{H}^1(X, \mathbf{G}') \to \check{H}^1(X, \mathbf{G}'') \xrightarrow{d^1} H^2(X, \mathbf{G}').$$

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