VECTOR BUNDLES OF RANK FOUR AND $\mathsf{A}_3 = \mathsf{D}_3$

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ABSTRACT. Over a scheme X with 2 invertible, we show that a version of "Pascal's rule" for vector bundles of rank 4 gives an explicit isomorphism between the moduli functors represented by projective homogeneous bundles for reductive group schemes of type A_3 and D_3. We exploit this to prove that a vector bundle $\mathscr{V}$ of rank 4 has a sub- or quotient line bundle if and only if the canonical symmetric bilinear form on $\bigwedge^2\mathscr{V}$ has a lagrangian subspace. Under additional hypotheses on X, we prove that this is equivalent to the vanishing of the Witt-theoretic Euler class $e(\mathscr{V}) \in W^4(X, \det \mathscr{V}^\vee)$.

1. INTRODUCTION

Let X be a scheme with 2 invertible and $\mathscr{V}$ a vector bundle (i.e. a locally free $\mathscr{O}_X$-module) of even rank $r = 2s$ on X. The middle exterior power $\bigwedge^s \mathscr{V}$ supports a canonical regular $\det(\mathscr{V})$-valued $(-1)^s$-symmetric bilinear form

$$\textstyle\bigwedge^s \mathscr{V} \otimes \bigwedge^s \mathscr{V} \xrightarrow{\wedge} \det \mathscr{V}$$

given by wedging. This operation defines a functor from the category (resp. stack) of vector bundles of even rank to the category (resp. stack) of regular line bundle-valued bilinear forms (see Proposition 2.3). When $\mathscr{V}$ is free, the middle exterior power form on $\bigwedge^s \mathscr{V}$ is hyperbolic. On the other hand, for example when $X = \mathbb{P}^4$ and $\mathscr{V} = \Omega^1_{\mathbb{P}^4}$ (see Walter [35]), the middle exterior power form need not even be metabolic. In this work, we give a general necessary condition for the middle exterior form to be metabolic, namely, that $\mathscr{V}$ has a sub- or quotient line bundle (which is locally a direct summand). This is related to a version of "Pascal's rule" (see Lemma 2.4) for vector bundles. When $\mathscr{V}$ has rank 4, our first main result (see Corollary 4.4) is that this condition is sufficient as well:

Theorem 1. *Let X be a scheme with 2 invertible and $\mathscr{V}$ be a vector bundle of rank 4 on X. Then $\mathscr{V}$ has a sub- or quotient line bundle if and only if the middle exterior power form on $\bigwedge^2 \mathscr{V}$ is metabolic.*

The proof involves the interplay between an explicit moduli interpretation and the (quite classical) geometry of the symmetric lagrangian grassmannian (see §2) as well as the exceptional isomorphism $\mathsf{A}_3 = \mathsf{D}_3$ (see Theorem 4.3). We then give (see Corollary 4.6) an interpretation of this result involving the Euler class (see §3) in Witt theory (in the sense of Fasel–Srinivas [19]).

Theorem 2. *Let X be a scheme with 2 invertible satisfying the "stable metabolicity" property. Let $\mathscr{V}$ a vector bundle of rank 4 on X. Then $\mathscr{V}$ has a sub- or quotient line bundle if and only if the Witt-theoretic Euler class $e(\mathscr{V}) \in W^4(X, \det \mathscr{V}^\vee)$ vanishes.*

This result follows from Theorem 1 by using explicit formulas for Euler classes found in Fasel–Srinivas [19, §2.4]. The "stable metabolicity" property is explored in §4.2, where a number of examples are provided.

Finally, in §4.3, we discuss necessary and sufficient conditions for the vanishing of the Euler class in Grothendieck–Witt (GW) theory in terms of the existence of free sub- or quotient line bundles (i.e. nonvanishing (co)sections). We prove the analogue of Theorem 1 (see Corollary 4.18) for vector bundles of rank 2, as well as partial results (see Proposition 4.19 and 4.20) concerning Euler classes of vector bundles of rank 2 and 4.

When 2 is not invertible, one should instead consider the natural *quadratic* form $\bigwedge^s \mathscr{V} \to \det \mathscr{V}$, see [27, II Prop. 10.12]. Likewise, replacing $\mathscr{V}$ by an Azumaya algebra $\mathscr{A}$ of degree $r = 2s$, one should consider the *canonical involution* on $\lambda^s \mathscr{A}$, see [27, Ch. II, §10.B] and Parimala–Sridharan [31]. The generalization to these contexts of the results presented here will be considered in the future.

2. Middle exterior power forms

Let X be a scheme with 2 invertible and $\mathscr{L}$ a line bundle (i.e. an invertible $\mathscr{O}_X$-module). An $\mathscr{L}$-*valued bilinear form* on X is a triple $(\mathscr{E}, b, \mathscr{L})$, where $\mathscr{E}$ is a vector bundle on X (i.e. locally free $\mathscr{O}_X$-module of constant finite rank) and $b : \mathscr{E} \otimes \mathscr{E} \to \mathscr{L}$ is an $\mathscr{O}_X$-module morphism. An $\mathscr{L}$-valued bilinear form is *symmetric* (resp. *alternating*) if b factors through the canonical epimorphism $\mathscr{E} \otimes \mathscr{E} \to S^2 \mathscr{E}$ (resp. $\mathscr{E} \otimes \mathscr{E} \to \bigwedge^2 \mathscr{E}$). As shorthand, a symmetric form will be called $(+1)$-symmetric and an alternating form (-1)-symmetric. An $\mathscr{L}$-valued bilinear form $(\mathscr{E}, b, \mathscr{L})$ is *regular* if the canonical adjoint $\psi_b : \mathscr{E} \to \mathscr{Hom}(\mathscr{E}, \mathscr{L})$ is an isomorphism of vector bundles on X. We will mostly dispense with the adjective "$\mathscr{L}$-valued".

Definition 2.1. Let $\mathscr{V}$ be a vector bundle of even rank $r = 2s$ on X. Denote by $\lambda^s \mathscr{V} = (\bigwedge^s \mathscr{V}, \wedge, \det \mathscr{V})$ the $\det \mathscr{V}$-valued *middle exterior power form*,

$$\bigwedge\nolimits^s \mathscr{V} \otimes \bigwedge\nolimits^s \mathscr{V} \xrightarrow{\wedge} \bigwedge\nolimits^r \mathscr{V} = \det \mathscr{V},$$

on X. It is $(-1)^s$-symmetric, regular, and of rank $n := \binom{r}{s}$.

2.1. **Torsorial interpretation.** The signed discriminant is defined for $\mathscr{L}$-valued forms of even rank (cf. Parimala–Sridharan [31, §4]). A *discriminant module* is a pair $(\mathscr{N}, n)$ consisting of a line bundle $\mathscr{N}$ and an $\mathscr{O}_X$-module isomorphism $n : \mathscr{N}^{\otimes 2} \to \mathscr{O}_X$. The isomorphism class of a discriminant module yields an element of $H^1_{\text{ét}}(X, \boldsymbol{\mu}_2)$ (cf. Milne [28, III §4] or Knus [26, III.3]). Applying the determinant functor to the adjoint morphism of a regular bilinear form $(\mathscr{E}, b, \mathscr{L})$ of even rank $n = 2m$ yields an isomorphism

$$\det \mathscr{E} \xrightarrow{\det \psi_b} \det \mathscr{Hom}(\mathscr{E}, \mathscr{L}) \xrightarrow{\text{can}} \mathscr{Hom}(\det \mathscr{E}, \mathscr{L}^{\otimes n}),$$

of vector bundles on X giving rise to a discriminant module $\operatorname{disc} b : (\det \mathscr{E} \otimes (\mathscr{L}^\vee)^{\otimes m})^{\otimes 2} \to \mathscr{O}_X$. As usual, we modify $\operatorname{disc} b$ by factor of $(-1)^m$ to define the *signed discriminant* module $d(\mathscr{E}, b, \mathscr{L})$. We denote by $\langle 1 \rangle$ the trivial discriminant module $\mathscr{O}_X^{\otimes 2} \to \mathscr{O}_X$, whose isomorphism class is the identity of the group $H^1_{\text{ét}}(X, \boldsymbol{\mu}_2)$.

The first fundamental property of a middle exterior power form is that it has trivial signed discriminant.

Lemma 2.2. *Let $\mathscr{V}$ be a vector bundle of rank $r = 2s$ on X. The middle exterior power form $\lambda^s \mathscr{V}$ has trivial discriminant. Moreover, there is a canonical choice of isomorphism $\zeta_{\mathscr{V}} : d(\lambda^s \mathscr{V}) \to \langle 1 \rangle$.*

Proof. When $\mathscr{V}$ is free with basis $e_1, \ldots, e_r$, define an $\mathscr{O}_X$-module isomorphism $\zeta : \det \bigwedge^s \mathscr{V} \to \det \mathscr{V}^{\otimes \frac{1}{2}\binom{r}{s}}$ by

$$\bigwedge_{1 \leq i_1 < \cdots < i_s \leq r} (e_{i_1} \wedge \cdots \wedge e_{i_s}) \quad \mapsto \quad (e_1 \wedge \cdots \wedge e_r)^{\otimes \frac{1}{2}\binom{r}{s}}.$$

A standard computation shows that ζ is $\mathbf{GL}(\mathscr{V})$-equivariant, thus does not depend on the choice of basis. Hence for a general vector bundle $\mathscr{V}$ or rank r, the map ζ patches over a Zariski open covering of X splitting $\mathscr{V}$. Finally, tensoring ζ by $(\det \mathscr{V}^\vee)^{\otimes \frac{1}{2}\binom{r}{s}}$ and scaling by $(-1)^m$ yields an $\mathscr{O}_X$-module morphism $\zeta_{\mathscr{V}} : d(\lambda^s \mathscr{V}) \to \langle 1 \rangle$, which is seen to be an isomorphism of discriminant modules by a local verification. $\square$

A *similarity* between bilinear forms $(\mathscr{E}, b, \mathscr{L})$ and $(\mathscr{E}', b', \mathscr{L}')$ is a pair (φ, λ) consisting of isomorphisms $\varphi : \mathscr{E} \to \mathscr{E}'$ and $\lambda : \mathscr{L} \to \mathscr{L}'$ such that either of the following (equivalent) diagrams,

(1)
$$
\begin{array}{ccc}
\mathscr{E} \otimes \mathscr{E} & \xrightarrow{\ b\ } & \mathscr{L} \\
{\scriptstyle \varphi \otimes \varphi} \downarrow & & \downarrow {\scriptstyle \lambda} \\
\mathscr{E}' \otimes \mathscr{E}' & \xrightarrow{\ b'\ } & \mathscr{L}'
\end{array}
\qquad
\begin{array}{ccc}
\mathscr{E} & \xrightarrow{\ \psi_b\ } & \mathscr{H}om(\mathscr{E}, \mathscr{L}) \\
{\scriptstyle \varphi} \downarrow & & \uparrow {\scriptstyle \lambda^{-1} \varphi^{\vee \mathscr{L}}} \\
\mathscr{E}' & \xrightarrow{\ \psi_{b'}\ } & \mathscr{H}om(\mathscr{E}', \mathscr{L}')
\end{array}
$$

of $\mathscr{O}_X$-modules commute, where $\lambda^{-1}\varphi^{\vee \mathscr{L}}(\psi) = \lambda^{-1} \circ \psi \circ \varphi$ on sections. Note that the commutativity of diagrams (1) takes on the familiar formula $b'(\varphi(v), \varphi(w)) = \lambda \circ b(v, w)$ on sections. A similarity transformation (φ, λ) is an *isometry* if $\mathscr{L} = \mathscr{L}'$ and λ is the identity map.

Denote by $\mathbf{Sim}(\mathscr{E}, b, \mathscr{L})$ the presheaf, on the large étale site $X_{\text{ét}}$, of similitudes of a regular $\mathscr{L}$-valued (± 1)-symmetric form $(\mathscr{E}, b, \mathscr{L})$. In fact, this is a sheaf and is representable by a smooth affine reductive group scheme over X. See Demazure–Gabriel [13, II.1.2.6, III.5.2.3]). Here we consider reductive group schemes whose fibers are not necessarily geometrically integral, in contrast to SGA 3 [12, III XIX.2]. When b is symmetric (resp. alternating), this is the *orthogonal similarity group* $\mathbf{GO}(\mathscr{E}, b, \mathscr{L})$ (resp. *symplectic similarity group* $\mathbf{GSp}(\mathscr{E}, b, \mathscr{L})$). Even though these sheaves of groups are representable by schemes over X, we will still think of them as sheaves of groups on $X_{\text{ét}}$.

Any similarity (φ, λ) between bilinear forms of even rank induces an isomorphism of signed discriminants $d(\varphi, \lambda) : d(\mathscr{E}, b, \mathscr{L}) \to d(\mathscr{E}', b', \mathscr{L}')$. When 2 is invertible on X, denote by $\mathbf{Sim}^+(\mathscr{E}, b, \mathscr{L})$ the sheaf kernel of the induced homomorphism $\mathbf{Sim}(\mathscr{E}, b, \mathscr{L}) \to \mathbf{Isom}(d(\mathscr{E}, b, \mathscr{L})) = \boldsymbol{\mu}_2$. If b is alternating, then $\mathbf{Sim}^+(\mathscr{E}, b, \mathscr{L}) = \mathbf{Sim}(\mathscr{E}, b, \mathscr{L})$; the proof in [27, III Prop. 12.3] can be adapted to $X_{\text{ét}}$. If b is symmetric (and 2 is invertible on X), then $\mathbf{Sim}^+(\mathscr{E}, b, \mathscr{L})$ is called the group of *proper* similarities; it is represented by a smooth connected reductive group scheme.

An *oriented bilinear form* $(\mathscr{E}, b, \mathscr{L}, \zeta)$ is a bilinear form of even rank together with an isomorphism $\zeta : \text{disc}(\mathscr{E}, b, \mathscr{L}) \to \langle 1 \rangle$ of discriminant modules. In particular, any oriented bilinear form is regular and has trivial signed discriminant. An *oriented similarity* between oriented bilinear forms $(\mathscr{E}, b, \mathscr{L}, \zeta)$ and $(\mathscr{E}', b', \mathscr{L}', \zeta')$ is a triple (φ, λ, ξ) where (φ, λ) is a similarity and $\xi : d(\mathscr{E}, b, \mathscr{L}) \to d(\mathscr{E}', b', \mathscr{L}')$ is an isomorphism of discriminant modules such that $\zeta' \circ \xi = \zeta$. When X is connected, every regular symmetric bilinear form with trivial discriminant has two oriented similarity classes.

For any vector bundle $\mathscr{F}$, any line bundle $\mathscr{L}$, and any $\epsilon \in \{\pm 1\}$, the *hyperbolic form* $H_{\mathscr{L}}^{\epsilon}(\mathscr{F})$ has underlying vector bundle $\mathscr{F} \oplus \mathscr{H}om(\mathscr{F}, \mathscr{L})$ and bilinear form given by $(v, f) \otimes (w, g) \mapsto f(w) + \epsilon g(v)$ on sections. It is ϵ-symmetric and regular. A hyperbolic form carries a canonical orientation $\zeta_{\mathscr{F}} : d(H_{\mathscr{L}}^{\epsilon}(\mathscr{F})) \to \langle 1 \rangle$. Every oriented ϵ-symmetric bilinear form of rank $n = 2m$ is, locally on $X_{\text{ét}}$, oriented similar to $H_{\mathscr{O}_X}^{\epsilon}(\mathscr{O}_X^m)$, see [2, Thm. 1.15]. For each $n = 2m$, put $\mathbf{Sim}_{m,m}^{\epsilon} = \mathbf{Sim}(H_{\mathscr{O}_X}^{\epsilon}(\mathscr{O}_X^m))$ and $\mathbf{Sim}_{m,m}^{+,\epsilon} = \mathbf{Sim}^+(H_{\mathscr{O}_X}^{\epsilon}(\mathscr{O}_X^m))$.

We now place the middle exterior power form into a functorial framework. For $r \geq 1$, denote by $\mathsf{VB}_r(X)$ the groupoid of vector bundles of rank r under isomorphism (this is isomorphic to the groupoid of $\mathbf{GL}_r$-torsors on $X_{\text{ét}}$). For even $n = 2m \geq 1$ and $\epsilon \in \{\pm 1\}$, denote by $\mathsf{BF}_n^{+,\epsilon}(X)$ the groupoid whose objects are oriented (line bundle-valued) ϵ-symmetric bilinear forms of rank n on X under oriented similarities (this is isomorphic to the groupoid of $\mathbf{Sim}_{m,m}^+$-torsors over $X_{\text{ét}}$). Denote by $\mathbf{VB}_r$ and $\mathbf{BF}_n^{+,\epsilon}$ the associated stacks over $X_{\text{ét}}$. There are canonical cartesian functors $\det : \mathbf{VB}_r \to \mathbf{Pic}$ and $\mu : \mathbf{BF}_n^{+,\epsilon} \to \mathbf{Pic}$ defined by the determinant and value line bundle, respectively, where $\mathbf{Pic}$ is the Picard stack on $X_{\text{ét}}$.

Proposition 2.3. *Let X be a scheme with 2 invertible, $r = 2s$ even, and $n = \binom{r}{s}$. Then the middle exterior power induces a cartesian functor $\lambda^s : \mathbf{VB}_r \to \mathbf{BF}_n^{+,(-1)^s}$ making the*

following diagram

$$\begin{array}{ccc}
\mathbf{VB}_r & \xrightarrow{\lambda^s} & \mathbf{BF}_n^{+,(-1)^s} \\
{\scriptstyle \det}\downarrow & & \downarrow{\scriptstyle \mu} \\
\mathbf{Pic} & = & \mathbf{Pic}
\end{array}$$

of stacks over $X_{\text{ét}}$ commute. In particular for each object $\mathscr{V}$ of $\mathsf{VB}_r(X)$, there's a canonical homomorphism of sheaves of groups $\lambda^s : \mathbf{GL}(\mathscr{V}) \to \mathbf{Sim}^+(\lambda^s\mathscr{V})$ on $X_{\text{ét}}$, making the following diagram

$$\begin{array}{ccc}
\mathbf{GL}(\mathscr{V}) & \xrightarrow{\lambda^s} & \mathbf{Sim}^+(\lambda^s\mathscr{V}) \\
{\scriptstyle \det}\downarrow & & \downarrow{\scriptstyle \mu} \\
\mathbf{G}_{\mathrm{m}} & = & \mathbf{G}_{\mathrm{m}}
\end{array}$$

of sheaves of groups on $X_{\text{ét}}$ commute.

Proof. For each $U \to X$ in $X_{\text{ét}}$ define $\lambda^s{:}\mathsf{VB}_r(U) \to \mathsf{BF}_n^{+,(-1)^s}(U)$ on objects by sending $\mathscr{V}$ to $(\lambda^s\mathscr{V}, \zeta_{\mathscr{V}})$, where $\zeta_{\mathscr{V}}$ is the orientation defined in Lemma 2.2. On morphisms, send an $\mathscr{O}_X$-isomorphism $\psi : \mathscr{V} \to \mathscr{V}'$ to the oriented similarity $\left(\bigwedge^s\psi, \bigwedge^r\psi, \bigwedge^n\left(\bigwedge^s\psi\right) \otimes \left(\bigwedge^r\psi^\vee\right)^{\otimes\frac{1}{2}\binom{r}{s}} \right)$. The only nontrivial thing to check is that this is indeed an oriented similarity! The fact that this gives rise to a cartesian functor of stacks follows from its nature as a tensorial construction. The fact that the diagrams commute follows directly from the definition. $\square$

2.2. **Some linear algebra.** Let X be a scheme and $\mathscr{V} \xrightarrow{f} \mathscr{N}$ be a morphism of $\mathscr{O}_X$-modules. For $s \geq 1$, the map $\mathscr{V}^{\otimes s} \to \mathscr{N} \otimes \mathscr{V}^{\otimes(s-1)}$ defined on sections by

$$v_0 \otimes \cdots \otimes v_{s-1} \mapsto \sum_{j=0}^{s-1} f(v_j) \otimes v_0 \otimes \cdots \otimes \hat{v}_j \otimes \cdots \otimes v_{s-1}$$

descends to the *contraction* map $d^{(s-1)}f : \bigwedge^s\mathscr{V} \to \mathscr{N}\otimes\bigwedge^{s-1}\mathscr{V}$. As usual, define $\bigwedge^0\mathscr{V} = \mathscr{O}_X$ so that $d^{(0)}f = f$. The composition

$$(2) \qquad \bigwedge^s\mathscr{V} \xrightarrow{d^{(s-1)}f} \mathscr{N}\otimes\bigwedge^{s-1}\mathscr{V} \xrightarrow{\mathrm{id}_{\mathscr{N}}\otimes d^{(s-2)}f} \mathscr{N}^{\otimes 2}\otimes\bigwedge^{s-2}\mathscr{V}$$

is zero. Hence for $r \geq 1$ we have an r-truncated *Koszul complex*

$$0 \to \bigwedge^r\mathscr{V} \xrightarrow{d^{(r-1)}f} \mathscr{N}\otimes\bigwedge^{r-1}\mathscr{V} \to \cdots \to \mathscr{N}^{\otimes(r-2)}\otimes\bigwedge^2\mathscr{V} \to \mathscr{N}^{\otimes(r-1)}\otimes\mathscr{V} \xrightarrow{\mathrm{id}_{\mathscr{N}^{\otimes(r-1)}}\otimes f} \mathscr{N}^{\otimes r} \to 0$$

of $\mathscr{O}_X$-modules. If $\mathscr{V}$ is a vector bundle of rank r and $\mathscr{N}$ is a line bundle then the r-truncated Koszul complex is denoted by $K(\mathscr{V}, f)$; it is exact if f is an epimorphism. Denoting by $\mathscr{W} = \ker f$ and $j : \mathscr{W} \to \mathscr{V}$ the inclusion, we have a complex

$$(3) \qquad 0 \to \bigwedge^s\mathscr{W} \xrightarrow{\wedge^s j} \bigwedge^s\mathscr{V} \xrightarrow{d^{(s-1)}f} \mathscr{N}\otimes\bigwedge^{s-1}\mathscr{V}$$

of $\mathscr{O}_X$-modules, which is exact under the conditions that $\mathscr{V}$ is a vector bundle, $\mathscr{N}$ is a line bundle, and f is an epimorphism. (Such an $\mathscr{N}$ will be called a *quotient line bundle* of $\mathscr{V}$.) The following linear algebra lemma is well-known.

Lemma 2.4 (Pascal's rule for vector bundles)**.** *Let*

$$0 \to \mathscr{W} \to \mathscr{V} \xrightarrow{f} \mathscr{N} \to 0,$$

be an exact sequence of vector bundles on X with $\mathscr{N}$ a line bundle. Then for any $s \geq 1$, the contraction map induces a short exact sequence,

$$(4) \qquad 0 \to \bigwedge^s\mathscr{W} \to \bigwedge^s\mathscr{V} \xrightarrow{d^{(s-1)}f} \mathscr{N}\otimes\bigwedge^{s-1}\mathscr{W} \to 0,$$

of vector bundles on X.

Proof. We only need to verify that the image of $d^{(s-1)}f$ in $\mathscr{N} \otimes \bigwedge^{s-1}\mathscr{V}$ is isomorphic to $\mathscr{N} \otimes \bigwedge^{s-1}\mathscr{W}$. Under our hypotheses, the Koszul complex (in particular, the complex (2)) is exact, hence the image of $d^{(s-1)}f$ is isomorphic to the kernel of $\mathrm{id}_{\mathscr{N}} \otimes d^{(s-2)}f$: $\mathscr{N} \otimes \bigwedge^{s-1}\mathscr{V} \to \mathscr{N}^{\otimes 2} \otimes \bigwedge^{s-2}\mathscr{V}$. Tensoring the degree $s-1$ version of (3) by $\mathscr{N}$, the kernel of $\mathrm{id}_{\mathscr{N}} \otimes d^{(s-2)}f$ is seen to be isomorphic to $\mathscr{N} \otimes \bigwedge^{s-1}\mathscr{W}$, proving the lemma. $\qquad\square$

Similarly, if $0 \to \mathscr{N}' \to \mathscr{V} \xrightarrow{g} \mathscr{W}' \to 0$ is an exact sequence of vector bundles on X with $\mathscr{N}'$ a line bundle (such an $\mathscr{N}'$ will be call a *sub-line bundle* of $\mathscr{V}$), then there is an induced short exact sequence

$$(5) \qquad 0 \to \mathscr{N}' \otimes \bigwedge^{s-1}\mathscr{W}' \to \bigwedge^s \mathscr{V} \xrightarrow{\wedge^s g} \bigwedge^s \mathscr{W}' \to 0,$$

by applying Lemma 2.4 to $\mathscr{V}^{\vee}$, then dualizing.

Remark 2.5. A trivial sub-line bundle $\mathscr{O}_X$ of $\mathscr{V}$ is also called a *nonvanishing global section*. If X is any noetherian affine scheme of (Krull) dimension d then Serre's splitting theorem guarantees a nonvanishing global section if $\mathscr{V}$ has rank $> d$. If X is a variety of dimension d over a field and $\mathscr{V}$ has rank $> d$, then $\mathscr{V}$ has a nonvanishing global section as soon as it is generated by global sections (cf. [23, II Exer. 8.2]). Furthermore, if X is projective over a field, then by a theorem of Serre, some twist $\mathscr{V}(n)$ is always generated by global sections (cf. [23, II Thm. 5.17]). Thus for a projective variety X of dimension d over a field, any vector bundle $\mathscr{V}$ of rank $> d$ on X has a sub-line bundle of the form $\mathscr{O}_X(-n)$ for $n \gg 0$.

A bilinear form $(\mathscr{E}, b, \mathscr{L})$ of rank $n = 2m$ on X is *metabolic* if there exists a vector subbundle $\mathscr{F} \to \mathscr{E}$ (i.e. locally a direct summand) of rank m such that the restriction of b to $\mathscr{F}$ is zero. Any choice of such $\mathscr{F}$ is called a *lagrangian*. For example, $\mathscr{F}$ is a lagrangian of the hyperbolic form $H^{\epsilon}_{\mathscr{L}}(\mathscr{F})$. An $\mathscr{O}_X$-submodule $\mathscr{F} \to \mathscr{E}$ is a lagrangian if and only if $\mathscr{F}$ is the kernel of $\psi_{b,\mathscr{F}} : \mathscr{E} \to \mathscr{H}om(\mathscr{F}, \mathscr{L})$ given by the composition of $\psi_b : \mathscr{E} \to \mathscr{H}om(\mathscr{E}, \mathscr{L})$ and the projection $\mathscr{H}om(\mathscr{E}, \mathscr{L}) \to \mathscr{H}om(\mathscr{F}, \mathscr{L})$. Thus a lagrangian $\mathscr{F}$ is also equivalent to an isomorphism of short exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{E} & \longrightarrow & \mathscr{E}/\mathscr{F} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \psi_b|_{\mathscr{F}}} & & \downarrow{\scriptstyle \psi_b} & & \downarrow{\scriptstyle \psi_{b,\mathscr{F}}} & & \\
0 & \longrightarrow & \mathscr{H}om(\mathscr{H}om(\mathscr{E}/\mathscr{F}, \mathscr{L}), \mathscr{L}) & \longrightarrow & \mathscr{H}om(\mathscr{E}, \mathscr{L}) & \longrightarrow & \mathscr{H}om(\mathscr{V}, \mathscr{L}) & \longrightarrow & 0
\end{array}
$$

of vector bundles, where the bottom sequence is the $\mathscr{L}$-dual of the top sequence. We will often refer to a self-dual short exact sequence, as above, by the term *lagrangian*. See Balmer [3], [4], and [5] for the general notion of metabolic form in the context of Grothendieck(-Witt) groups of triangulated and exact categories with duality.

Proposition 2.6. *Let $\mathscr{V}$ be a vector bundle of rank $r = 2s$ and $\mathscr{N}$, $\mathscr{N}'$ be line bundles on X.*

(1) *For an exact sequence of vector bundles of the form*

$$0 \to \mathscr{W} \to \mathscr{V} \xrightarrow{f} \mathscr{N} \to 0,$$

the associated exact sequence (4) is a lagrangian of $\lambda^s\mathscr{V}$.

(2) *For an exact sequence of vector bundles of the form*

$$0 \to \mathscr{N}' \to \mathscr{V} \xrightarrow{g} \mathscr{W}' \to 0,$$

the associated exact sequence (5) is a lagrangian of $\lambda^s\mathscr{V}$.

(3) *If $\mathscr{V} \cong \mathscr{N} \oplus \mathscr{W}$, then $\lambda^s\mathscr{V}$ is isometric to the hyperbolic form $H^{(-1)^s}_{\det \mathscr{V}}(\bigwedge^s\mathscr{W})$.*

Proof. A linear algebra exercise using the description of lagrangians as self-dual short exact sequences. $\qquad\square$

Corollary 2.7. *If $\mathscr{V}$ has a sub- or quotient line bundle then $\lambda^s\mathscr{V}$ is metabolic.*

The statement of Corollary 2.7 can be also seen indirectly as a consequence of the Whitney formula for the Euler class in Grothendieck–Witt theory together with explicit formulas for its computation (cf. Proposition 3.2). We will examine in further detail the connection between middle exterior power forms and the Euler class in §3.

As a consequence, we give a generalization to line bundle-valued forms over schemes of a result of Knus [26, V Prop. 5.1.10]. A line bundle-valued quadratic form of even rank and trivial discriminant has an associated *refined Clifford invariant* in $H^2_{\text{ét}}(X, \boldsymbol{\mu}_2)$ constructed in [2, §2.8] (denoted by gc^+). Two bilinear forms are *projectively similar* if they are similar up to tensoring by a regular line bundle-valued bilinear form of rank 1.

Corollary 2.8. *Let X be a scheme with 2 invertible and with the property that any vector bundle of rank 4 has a sub- or quotient line bundle. Then any line bundle-valued symmetric bilinear form of rank 6 over X with trivial discriminant and trivial refined Clifford invariant is metabolic.*

Proof. Any regular line bundle-valued symmetric bilinear form $(\mathscr{E}, q, \mathscr{L})$ of rank 6 with trivial discriminant is isometric to a reduced pfaffian form (see Bichsel–Knus [8, Thm. 5.2]). Any reduced pfaffian form of rank 6 with trivial refined Clifford invariant is projectively similar to $\lambda^2 \mathscr{V}$ for some $\mathscr{V}$ of rank 4 (see [2, Prop. 2.21]). The hypotheses and Corollary 2.7 imply that $\lambda^2 \mathscr{V}$ is metabolic, hence $(\mathscr{E}, q, \mathscr{L})$ is metabolic. $\square$

By Remark 2.5, any X of dimension ≤ 3 that is either noetherian affine or projective over a field satisfies the hypotheses of Corollary 2.8.

Remark 2.9. For a general vector bundle $\mathscr{V}$, let $p : \mathbb{P}(\mathscr{V}) = \mathbf{Proj}\, S(\mathscr{V}) \to X$ be the associated projective bundle. Due to the universal quotient line bundle $p^* \mathscr{V} \to \mathscr{O}_{\mathbb{P}(\mathscr{V})}(1)$, the form $p^*(\lambda^s \mathscr{V})$ is always metabolic, with a canonical choice of lagrangian given by Proposition 2.6. Thus any section $\sigma : X \to \mathbb{P}(\mathscr{V})$ of p (i.e. $\sigma \in \mathbb{P}(\mathscr{V})(X)$) gives rise to a lagrangian of $\lambda^s \mathscr{V} = \sigma^* p^*(\lambda^s \mathscr{V})$ via pullback. But each $\sigma \in \mathbb{P}(\mathscr{V})(X)$ corresponds to a quotient line bundle $\mathscr{V} = \sigma^* p^* \mathscr{V} \to \sigma^* \mathscr{O}_{\mathbb{P}(\mathscr{V})}(1)$, providing another way of understanding Proposition 2.6, see §2.3 for more details. In particular, $\lambda^s \mathscr{V}$ is metabolic if $\mathbb{P}(\mathscr{V})(X) \neq \varnothing$.

2.3. Lagrangian grassmannian.
The lagrangian grassmannian (or grassmannian of maximal isotropic subspaces) of a bilinear form is a well-studied object. Our perspective is to consider these spaces as moduli functors and as projective homogeneous space fibrations.

A *polarization* on a scheme X is an isomorphism class of $\mathscr{O}_X$-module $\mathscr{F}$. A morphism $(f, \varphi) : (X, \mathscr{F}) \to (Y, \mathscr{G})$ of polarized schemes is a morphism $f : X \to Y$ of schemes together with a morphism of $\mathscr{O}_Y$-modules $\varphi : \mathscr{G} \to f_* \mathscr{F}$.

Let $(\mathscr{E}, b, \mathscr{L})$ be a regular ϵ-symmetric bilinear form of even rank $n = 2m$ on X. Denote by $\Lambda \mathbb{G}(\mathscr{E}, b, \mathscr{L})$ its moduli space of lagrangians, called the *lagrangian grassmannian*, and $\mathscr{T}$ the universal lagrangian subbundle. The polarized X-scheme $(\Lambda \mathbb{G}(\mathscr{E}, b, \mathscr{L}), \mathscr{T})$ represents the functor

$$u : U \to X \quad \mapsto \quad \left\{ \text{lagrangians } \mathscr{F} \xrightarrow{\varphi} u^* \mathscr{E} \text{ of } u^*(\mathscr{E}, b, \mathscr{L}) \right\}$$

where $\mathscr{F} \xrightarrow{\varphi} u^* \mathscr{E}$ and $\mathscr{F}' \xrightarrow{\varphi'} u^* \mathscr{E}$ are considered equivalent if and only if there exists an $\mathscr{O}_U$-module isomorphism $\psi : \mathscr{F} \to \mathscr{F}'$ such that $\varphi = \varphi' \circ \psi$. Note that if $(\mathscr{E}, b, \mathscr{L})$ is not a split metabolic, then an isomorphism $\mathscr{F} \to \mathscr{F}'$ of lagrangians cannot necessarily be extended to an isometry of $(\mathscr{E}, b, \mathscr{L})$.

The stabilizer subgroup $\mathbf{P}_{\mathscr{F}} = \mathbf{P}_{\mathscr{F}}(\mathscr{E}, b, \mathscr{L}) \to \mathbf{Sim}(\mathscr{E}, b, \mathscr{L})$ of any lagrangian $\mathscr{F} \to \mathscr{E}$ is a parabolic subgroup. The morphism $\Lambda \mathbb{G}(\mathscr{E}, b, \mathscr{L}) \to X$ is a projective homogeneous space for the group scheme $\mathbf{Sim}(\mathscr{E}, b, \mathscr{L})$, i.e. a moduli space of parabolic subgroups of $\mathbf{Sim}(\mathscr{E}, b, \mathscr{L})$ of a given type. It has geometrically integral fibers when $\epsilon = -1$ (the C_m case). When $\epsilon = 1$ (the D_m case), it's Stein factorization is of the form $\Lambda \mathbb{G}(\mathscr{E}, b, \mathscr{L}) \xrightarrow{\pi} Z \xrightarrow{f} X$ where π is smooth projective and with geometrically integral fibers and f is the étale double cover of X given by the signed discriminant $d(\mathscr{E}, b, \mathscr{L})$. Indeed, by going to fibers over X, this is a consequence of the corresponding fact over fields; see [14, §85]) or [24, Prop. 3.3] for a more global argument. When the signed discriminant is trivial, $\Lambda \mathbb{G}(\mathscr{E}, b, \mathscr{L})$ has two connected components over X, each of which is a $\mathbf{Sim}^+(\mathscr{E}, b, \mathscr{L})$ orbit. In this case, two lagrangians $\mathscr{F} \to \mathscr{E}$ and $\mathscr{F}' \to \mathscr{E}$ correspond to points in the same connected component

if and only if $\dim_{\kappa(x)}(\mathscr{F}(x) \cap \mathscr{F}'(x)) \equiv m \mod 2$ for each point x of X, where $\mathscr{F}(x)$ is the fiber over the residue field $\kappa(x)$, see Bourbaki [9, §5 Exer. 18d], Mumford [30], or Fulton [20].

Recall that for a vector bundle $\mathscr{V}$ of rank r on X, if $\mathscr{O}_{\mathbb{P}(\mathscr{V})/X}(1)$ is the universal quotient line bundle on $\mathbb{P}(\mathscr{V}) \to X$, then the polarized X-scheme $(\mathbb{P}(\mathscr{V}), \mathscr{O}_{\mathbb{P}(\mathscr{V})/X}(1))$ represents the moduli functor

$$u : U \to X \quad \mapsto \quad \{\text{quotient line bundles } u^*\mathscr{V} \xrightarrow{f} \mathscr{N}\}$$

where $u^*\mathscr{V} \xrightarrow{f} \mathscr{N}$ and $u^*\mathscr{V} \xrightarrow{f'} \mathscr{N}'$ are considered equivalent if and only if there exists an isomorphism $\mu : \mathscr{N} \to \mathscr{N}'$ such that $f' = \mu \circ f$. By dualizing, we can view $\mathbb{P}(\mathscr{V}^\vee)$ as representing the moduli space of sub-line bundles of $\mathscr{V}$.

The projective bundle $\mathbb{P}(\mathscr{V}) \to X$ is a projective homogeneous space for the reductive group scheme $\mathbf{GL}(\mathscr{V})$ of type A_{r-1}. The stabilizer subgroup $\mathbf{P}_{\mathscr{W}} = \mathbf{P}_{\mathscr{W}}(\mathscr{V}) \to \mathbf{GL}(\mathscr{V})$ of any vector subbundle $\mathscr{W} \to \mathscr{V}$ with line bundle quotient is a parabolic subgroup.

Theorem 2.10. *Let X be a scheme with 2 invertible and $\mathscr{V}$ be a vector bundle of even rank $r = 2s$ on X. There exist canonical X-morphisms of schemes*

$$\Phi_{\mathscr{V}} : \mathbb{P}(\mathscr{V}) \to \Lambda\mathbb{G}(\lambda^s \mathscr{V}), \qquad \Phi'_{\mathscr{V}} : \mathbb{P}(\mathscr{V}^\vee) \to \Lambda\mathbb{G}(\lambda^s \mathscr{V}),$$

which realize, on the level of moduli functors, the maps

$$\left(0 \to \mathscr{W} \to \mathscr{V} \xrightarrow{f} \mathscr{N} \to 0\right) \quad \mapsto \quad \left(0 \to \bigwedge\nolimits^s \mathscr{W} \to \bigwedge\nolimits^s \mathscr{V} \xrightarrow{df} \mathscr{N} \otimes \bigwedge\nolimits^{s-1} \mathscr{W} \to 0\right)$$

$$\left(0 \to \mathscr{N}' \to \mathscr{V} \xrightarrow{g} \mathscr{W}' \to 0\right) \quad \mapsto \quad \left(0 \to \mathscr{N}' \otimes \bigwedge\nolimits^{s-1} \mathscr{W}' \to \bigwedge\nolimits^s \mathscr{V} \xrightarrow{\wedge^s g} \bigwedge\nolimits^s \mathscr{W}' \to 0\right)$$

defined by Proposition 2.6, respectively (here $\mathscr{N}$ and $\mathscr{N}'$ are line bundles). Furthermore, $\Phi_{\mathscr{V}}$ and $\Phi'_{\mathscr{V}}$ map to different connected components over X if and only if s is a power of 2.

Proof. Since the Koszul complex is functorial, the maps defined by Proposition 2.6 define morphisms of moduli functors. By Yoneda's lemma, these are representable by X-morphism of the representing moduli schemes.

Now we prove the final claim. If s is odd (the C_m case) then $\Lambda\mathbb{G}(\lambda^s \mathscr{V}) \to X$ is connected, so we need only consider the case when s is even (the D_m case). The fibral criterion for two lagrangians to be in the same connected component reduces us to linear algebra considerations. Let V be a vector space of dimension $r = 2s$ (with s even) over a field k, let $W, W' \subset V$ be subspaces of codimension 1, and choose a splitting $V = W' \oplus L'$. The codimension of $W \cap W' \subset V$ is either 1 or 2. In the former case, $W = W'$ and we have that $\bigwedge^s W \cap L' \otimes \bigwedge^{s-1} W' = \{0\}$ since $W \cap L' = \{0\}$. As the rank of an intersection of lagrangians has a well-defined parity, we see that $\dim_k\left(\bigwedge^s W \cap L' \otimes \bigwedge^{s-1} W'\right)$ is always even. Finally, we use the fact (cf. [27, II Lemma 10.29]) that $\dim_k(\lambda^s V) = \binom{r}{s} \equiv 2 \mod 4$ if and only if s is a power of 2. Thus $\dim_k\left(\bigwedge^s W \cap L' \otimes \bigwedge^{s-1} W'\right) \not\equiv \frac{1}{2}\binom{r}{s} \mod 2$ if and only if s is a power of 2 and the claim follows by appealing to the above fibral criterion. $\qquad\square$

3. The Euler class in Grothendieck–Witt theory

Let X be a scheme with 2 invertible. Modeled on classical treatments of the Koszul complex (cf. [10, §1.6]), Balmer, Gille, and Nenashev [7], [21] introduce the *Euler class* $e(\mathscr{V}) \in GW^r(X, \det \mathscr{V}^\vee)$ of a vector bundle $\mathscr{V}$ of rank r on X. Also see Fasel [16, §2.4]. Here we mostly follow the treatment in Fasel–Srinivas [19, §2.4].

For a scheme X with 2 invertible and $\mathscr{N}$ a line bundle on X, the r-shifted $\mathscr{N}$-twisted derived Grothendieck–Witt groups $GW^r(X, \mathscr{N})$ were introduced by Balmer [3], [4], [5] and Walter [36]. For $\epsilon \in \{\pm 1\}$, the classical Grothendieck–Witt group $GW^\epsilon(X, \mathscr{N})$ of regular ϵ-symmetric bilinear forms on X was introduced by Knebusch [25]. The map taking a regular ϵ-symmetric bilinear form $(\mathscr{E}, b, \mathscr{N})$ to the complex $\mathscr{E}[s]$ consisting of $\mathscr{E}$ concentrated in degree s together with the natural $2s$-shifted $\mathscr{N}$-twisted symmetric isomorphism induced by b gives rise to a group isomorphism $GW^{(-1)^s}(X, \mathscr{N}) \to GW^{2s}(X, \mathscr{N})$ for any s, see Walter [36, Thm. 6.1].

3.1. **Euler classes and middle exterior power forms.** Let $\mathscr{V}$ be a vector bundle of rank r and $\mathscr{N}$ a line bundle on a scheme X. Given an $\mathscr{O}_X$-module morphism $\mathscr{V} \xrightarrow{f} \mathscr{N}$, consider the (twisted) Koszul complex $K(\mathscr{V}, f)$

$$0 \to (\mathscr{N}^\vee)^{\otimes(r-1)} \otimes \bigwedge\nolimits^r \mathscr{V} \to \cdots \to \mathscr{N}^\vee \otimes \bigwedge\nolimits^2 \mathscr{V} \xrightarrow{d^{(1)}f} \mathscr{V} \xrightarrow{f} \mathscr{N} \to 0$$

of vector bundles on X, where $\mathscr{N}$ is in degree 0. Then $\mathscr{N}^\vee \otimes K(\mathscr{V}, f)$ is isomorphic to a standard Koszul complex $K(\mathscr{N}^\vee \otimes \mathscr{V}, \mathrm{ev} \circ (\mathrm{id} \otimes f))$ associated to the cosection $\mathscr{N}^\vee \otimes \mathscr{V} \to \mathscr{O}_X$. The perfect pairings $\bigwedge^j \mathscr{V} \otimes \bigwedge^{r-j} \mathscr{V} \to \det \mathscr{V}$ together with certain sign conventions give rise to a symmetric isomorphism of complexes $\Phi : K(\mathscr{V}, f) \to K(\mathscr{V}, f)^\sharp$, where $(-)^\sharp = \mathscr{H}om\big(-, (\mathscr{N}^\vee)^{\otimes(r-2)} \otimes \det \mathscr{V}\big)[r]$, see Fasel–Srinivas [19, §2.4] or Fasel [16, §2.4]. Thus the pair $(K(\mathscr{V}, f), \Phi)$ gives rise to a class in $GW^r(X, (\mathscr{N}^\vee)^{\otimes(r-2)} \otimes \det \mathscr{V})$.

Consider the associated affine bundle $p : \mathbb{V}(\mathscr{V}) = \mathbf{Spec}\, S(\mathscr{V}) \to X$ and its zero section $s : X \to \mathbb{V}(\mathscr{V})$. Then $p^* \mathscr{V}^\vee$ has a canonical "evaluation" morphism $f : p^* \mathscr{V}^\vee \to \mathscr{O}_{\mathbb{V}(\mathscr{V})}$ with cokernel $s_* \mathscr{O}_X$. There's an associated Koszul complex $K(p^* \mathscr{V}^\vee, f)$

$$0 \to \bigwedge\nolimits^r p^* \mathscr{V}^\vee \to \cdots \to \bigwedge\nolimits^2 p^* \mathscr{V}^\vee \xrightarrow{d^{(1)}f} p^* \mathscr{V}^\vee \xrightarrow{f} \mathscr{O}_{\mathbb{V}(\mathscr{V})} \to 0$$

of vector bundles on $\mathbb{V}(\mathscr{V})$. The canonical morphism of sheaves $s^\sharp : \mathscr{O}_{\mathbb{V}(\mathscr{V})} \to s_* \mathscr{O}_X$ defines an isomorphism $K(p^* \mathscr{V}^\vee, f) \to s_* \mathscr{O}_X$ in the derived category. The zero section defines a pullback map

$$s^* : GW^r(\mathbb{V}(\mathscr{V}), p^* \det \mathscr{V}^\vee) \to GW^r(X, \det \mathscr{V}^\vee)$$

on Grothendieck–Witt groups.

Definition 3.1. Let $\mathscr{V}$ be a vector bundle of rank r on X. The *Euler class* $e(\mathscr{V}) \in GW^r(X, \det \mathscr{V}^\vee)$ is defined to be $s^*(K(p^* \mathscr{V}^\vee, f), \Phi)$. The *middle exterior power class* $\lambda(\mathscr{V}) \in GW^r(X, \det \mathscr{V})$ is defined to be the Grothendieck–Witt group class $\lambda^s \mathscr{V}[r]$ if $r = 2s$ is even and to be 0 if r is odd.

Proposition 3.2 (Fasel–Srinivas [19, Prop. 14, 21]). *Let X be a scheme with 2 invertible and $\mathscr{V}$ a vector bundle of rank r on X.*

(1) *(Explicit formula for Euler classes) In $GW^r(X, \det \mathscr{V}^\vee)$, we have the following formula*

$$e(\mathscr{V}) = \langle (-1)^{s(s-1)/2} \rangle \otimes \lambda(\mathscr{V}^\vee) + H_{\det \mathscr{V}^\vee} \left(\sum_{j=0}^{\lfloor (s-1)/2 \rfloor} (-1)^j \big[\bigwedge\nolimits^j \mathscr{V}^\vee \big] \right).$$

(2) *(Whitney sum formula for Euler classes) If $0 \to \mathscr{W} \to \mathscr{V} \to \mathscr{N} \to 0$ is an exact sequence of vector bundles, then $e(\mathscr{V}) = e(\mathscr{W}) \cdot e(\mathscr{N})$ under the multiplication in Grothendieck–Witt groups induced by $\det \mathscr{W}^\vee \otimes \det \mathscr{N}^\vee \to \det \mathscr{V}^\vee$.*

Remark 3.3. Tensoring with $\det \mathscr{V}^\vee$ — actually multiplication in the Grothendieck–Witt group by the rank one bilinear form $(\det \mathscr{V}^\vee, \otimes, (\det \mathscr{V}^\vee)^{\otimes 2})$ — yields a canonical isomorphism of groups $GW^r(X, \det \mathscr{V}) \to GW^r(X, \det \mathscr{V}^\vee)$ via the evaluation morphism $\det \mathscr{V} \otimes (\det \mathscr{V}^\vee)^{\otimes 2} \to \det \mathscr{V}^\vee$. Under this isomorphism the class of $\lambda^s \mathscr{V}$ is mapped to the class of $\lambda^s \mathscr{V}^\vee$. Also note that $\lambda^s \mathscr{V}$ is metabolic if and only if $\langle \pm 1 \rangle \otimes \lambda^s \mathscr{V}$ is metabolic.

The following corollary can be viewed as a generalization of Calmès–Hornbostel [11, Rem. 7.5].

Corollary 3.4. *If $\mathscr{V}$ has a sub- or quotient line bundle then $e(\mathscr{V})$ is metabolic.*

Proof. Combine the explicit formula for Euler classes (cf. Proposition 3.2) and Corollary 2.7. Another proof employs the Whitney sum formula for Euler classes (cf. Proposition 3.2) noting that the product of metabolic classes is metabolic. $\square$

3.2. **Euler classes and quotient line bundles.** For any line bundle $\mathscr{N}$, Karoubi periodicity gives rise to the exact sequence

(6) $$GW^{r-1}(X, \mathscr{N}) \xrightarrow{f} K_0(X) \xrightarrow{H_{\mathscr{N}}} GW^r(X, \mathscr{N}) \to W^r(X, \mathscr{N}) \to 0$$

where f is the forgetful map, see Walter [36, Thm. 2.6]. If a vector bundle $\mathscr{V}$ of even rank $r = 2s$ on X has a sub- or quotient line bundle $\mathscr{L}$, then by Propositions 2.6 and 3.2, the Euler class $e(\mathscr{V})$ is hyperbolic:

(7) $$e(\mathscr{V}) = H_{\det \mathscr{V}^\vee}\left((-1)^s\big[\textstyle\bigwedge^s \mathscr{W}^\vee\big] + \sum_{j=0}^{s-1}(-1)^j\big[\textstyle\bigwedge^j \mathscr{V}^\vee\big]\right),$$

where $\mathscr{W}$ is the vector bundle complementary to $\mathscr{L}$. We can give a useful representation of this class. For any vector bundle $\mathscr{W}$, denote by

$$\textstyle\bigwedge \mathscr{W} = \sum_{j=0}^{\mathrm{rk}\,\mathscr{W}}(-1)^j\big[\textstyle\bigwedge^j \mathscr{W}^\vee\big]$$

in $K_0(X)$. Recall that $\bigwedge \mathscr{W} = s^* s_* \mathscr{O}_X$, where $s : X \to \mathbb{V}(\mathscr{W})$ is the zero section and s_* and s^* are the associated pushforward and pullback maps on K_0.

Proposition 3.5. *Let $\mathscr{V}$ be a vector bundle of rank r on X and suppose that $0 \to \mathscr{W} \to \mathscr{V} \to \mathscr{L} \to 0$ is a short exact sequence of vector bundles with $\mathscr{L}$ a line bundle. Then we have*

$$e(\mathscr{V}) = H_{\det \mathscr{V}^\vee}\left(\textstyle\bigwedge \mathscr{W}\right)$$

in $GW^r(X, \det \mathscr{V}^\vee)$.

Proof. The proof is a straightforward calculation using the explicit formula for Euler classes (cf. Proposition 3.2), Lemma 2.4, and the fact that, for all $j \geq 1$,

$$\big[\mathscr{L}^\vee \otimes \textstyle\bigwedge^{j-1}\mathscr{W}^\vee\big] - \big[\textstyle\bigwedge^{r-j}\mathscr{W}^\vee\big]$$

is in the kernel of the hyperbolic map $H_{\det \mathscr{V}^\vee} : K_0(X) \to GW^r(X, \det \mathscr{V}^\vee)$. $\qquad\square$

In the case when $\mathscr{V}$ has odd rank, a formula similar to Proposition 3.5 appears in Fasel [18, Thm. 10.1]. For future reference, we record the equality

(8) $$\det \mathscr{W} \otimes \textstyle\bigwedge \mathscr{W} = (-1)^{\mathrm{rk}\,\mathscr{W}} \textstyle\bigwedge \mathscr{W}^\vee$$

in $K_0(X)$.

4. Rank four vector bundles

In this section, we apply the above general results to the specific case of vector bundles of rank four. In this special situation, we are helped by the exceptional isomorphism $A_3 = D_3$.

4.1. **Middle exterior forms.** Let X be a scheme with 2 invertible. By Proposition 2.3, the middle exterior power functor gives rise to a canonical homomorphism $\lambda^2 : \mathbf{GL}(\mathscr{V}) \to \mathbf{GSO}(\lambda^2 \mathscr{V})$. When $\mathscr{V}$ has rank four, this homomorphism is an isogeny.

Proposition 4.1. *Let X be a scheme with 2 invertible and $\mathscr{V}$ be a vector bundle of rank 4 on X. There's a short exact sequence*

$$1 \to \boldsymbol{\mu}_2 \to \mathbf{GL}(\mathscr{V}) \xrightarrow{\lambda^2} \mathbf{GSO}(\lambda^2 \mathscr{V}) \to 1$$

of sheaves of groups on $X_{\text{ét}}$.

Proof. The only thing to check is that λ^2 is an epimorphism on $X_{\text{ét}}$, which is known, cf. Knus [26, V.5.6]. $\qquad\square$

Remark 4.2. The coboundary map $H^1_{\text{ét}}(X, \mathbf{GSO}(\lambda^2 \mathscr{V})) \to H^2_{\text{ét}}(X, \boldsymbol{\mu}_2)$ associated to the exact sequence in Proposition 4.1 has the following interpretation: an (oriented) regular line bundle-valued symmetric bilinear form of rank 6 is mapped to its refined Clifford invariant constructed in [2, §2.8], cf. Corollary 2.8.

The D_3 version of Theorem 2.10 is more precise, yielding a stronger version of Corollary 2.7 as follows.

Theorem 4.3. *Let X be a scheme with 2 invertible and $\mathscr{V}$ be a vector bundle of rank 4 on X. Then*

$$\Phi_{\mathscr{V}} \sqcup \Phi'_{\mathscr{V}} : \mathbb{P}(\mathscr{V}) \sqcup \mathbb{P}(\mathscr{V}^{\vee}) \to \Lambda\mathbb{G}(\lambda^2\mathscr{V})$$

is an isomorphism of X-schemes.

Proof. One can argue, by passing to fibers, using the classical case over fields. We prefer to argue directly as follows. Let $\mathbf{P} \hookrightarrow \mathbf{GL}_4$ be the parabolic subgroup given by the stabilizer of $\mathscr{O}_X^3 \subset \mathscr{O}_X^4$ and let $\mathbf{Q} \hookrightarrow \mathbf{GSO}_{3,3}$ be the parabolic subgroup corresponding to the associated choice of oriented lagrangian $\bigwedge^2 \mathscr{O}_X^3$ of $\lambda^2 \mathscr{O}_X^4 \cong H_{\mathscr{O}_X}(\mathscr{O}_X^3)$. Then upon restricting the morphism λ^2, we have a commutative diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \boldsymbol{\mu}_2 & \longrightarrow & \mathbf{P} & \xrightarrow{\ \lambda^2\ } & \mathbf{Q} & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \boldsymbol{\mu}_2 & \longrightarrow & \mathbf{GL}_4 & \xrightarrow{\ \lambda^2\ } & \mathbf{GSO}_{3,3} & \longrightarrow & 1
\end{array}
$$

of groups schemes on X extending Proposition 4.1. Similarly, we have a epimorphism of (right) torsors

$$\mathbf{Isom}(\mathscr{O}_X^4, \mathscr{V}) \xrightarrow{\ \lambda^2\ } \mathbf{Sim}^+(\lambda^2\mathscr{O}_X^4, \lambda^2\mathscr{V})$$

equivariant for the corresponding homomorphism of group schemes. Then in this situation, we have an induced commutative diagram of X-scheme isomorphisms

$$
\begin{array}{ccc}
\mathbf{Isom}(\mathscr{O}_X^4, \mathscr{V})/\mathbf{P} & \xrightarrow{\ \sim\ } & \mathbf{Sim}^+(\lambda^2\mathscr{O}_X^4, \lambda^2\mathscr{V})/\mathbf{Q} \\
\wr\downarrow & & \downarrow\wr \\
\mathbb{P}(\mathscr{V}) & \xrightarrow{\ \Phi_{\mathscr{V}}\ } & \Lambda\mathbb{G}(\lambda^2\mathscr{V})^{\circ}
\end{array}
$$

where $\Lambda\mathbb{G}(\lambda^2\mathscr{V})^{\circ}$ is the connected component containing the image of $\Phi_{\mathscr{V}}$ and the vertical isomorphism are a consequence of SGA 3 [12, III XXVI.3 Lemma 3.2], since these projective homogeneous schemes are moduli spaces of parabolic subgroups of the associated groups.

Note that if $\mathbf{GL}(\mathscr{V})$ actually had a parabolic subgroup $\mathbf{P}_{\mathscr{W}}$ of the same type as $\mathbf{P}$ (i.e. there exists an exact sequence $0 \to \mathscr{W} \to \mathscr{V} \to \mathscr{L} \to 0$ with $\mathscr{L}$ a line bundle), then the above argument can be summarized with the following commutative diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \boldsymbol{\mu}_2 & \longrightarrow & \mathbf{P}_{\mathscr{W}} & \xrightarrow{\ \lambda^2\ } & \mathbf{P}_{\lambda^2\mathscr{W}} & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \boldsymbol{\mu}_2 & \longrightarrow & \mathbf{GL}(\mathscr{V}) & \xrightarrow{\ \lambda^2\ } & \mathbf{GSO}(\lambda^2\mathscr{V}) & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathbb{P}(\mathscr{V}) & \xdashrightarrow{\ \Phi_{\mathscr{V}}\ } & \Lambda\mathbb{G}(\lambda^2\mathscr{V})^{\circ} & &
\end{array}
$$

of X-scheme, where the top two rows are short exact sequences of group schemes.

So far, we've shown that $\Phi_{\mathscr{V}}$ is an isomorphism onto a connected component. The same argument can be used for $\Phi'_{\mathscr{V}}$ (using the dual parabolic subgroup of $\mathbf{P}$). Since $\Lambda\mathbb{G}(\lambda^2\mathscr{V}) \to X$ has two connected X-components (since the discriminant of $\lambda^2\mathscr{V}$ is trivial) and $\Phi_{\mathscr{V}}$ and $\Phi'_{\mathscr{V}}$ map to different components, we are done. $\qquad\square$

Corollary 4.4. *Let X be a scheme with 2 invertible and $\mathscr{V}$ a vector bundle of rank 4 on X. Then $\mathscr{V}$ has a sub- or quotient line bundle if and only if $\lambda^2\mathscr{V}$ is metabolic.*

Of course, if $r = 2s$ is even and $\lambda^s\mathscr{V}$ is metabolic, then the Witt class $\lambda(\mathscr{V}) \in W^r(X, \det\mathscr{V})$ vanishes (see Remark 3.3). In general, the converse — that if $\lambda^s\mathscr{V}$ is stably metabolic (i.e. it's Witt class $\lambda(\mathscr{V})$ is trivial) then it is metabolic — may not hold. However, under certain hypotheses on X, which we shall now outline, the converse does indeed hold.

4.2. **Stably metabolic forms.** We introduce the following properties of an exact category $C = (C, {}^\sharp, \varpi)$ with duality and 2 invertible, see Quebbemann–Scharlau–Schulte [32], [33], Knus [26, II], Balmer [6, §1.1.2], or Walter [36, §6] for definitions.

Definition 4.5. The *stable metabolicity* (sM) property: given ϵ-symmetric objects $(\mathscr{E}, b)$ and $(\mathscr{M}, h)$ of C with $(\mathscr{M}, h)$ metabolic, if $(\mathscr{E}, b) \perp (\mathscr{M}, h)$ is metabolic then $(\mathscr{E}, b)$ is metabolic. Equivalently, $(\mathscr{E}, b)$ is metabolic if it has trivial Witt class.

The *Witt cancellation* (Wc) property: given ϵ-symmetric objects $(\mathscr{E}_1, b_1)$, $(\mathscr{E}_2, b_2)$, and $(\mathscr{F}, b)$ of C, if $(\mathscr{E}_1, b_1) \perp (\mathscr{F}, b) \cong (\mathscr{E}_2, b_2) \perp (\mathscr{F}, b)$ then $(\mathscr{E}_1, b_1) \cong (\mathscr{E}_2, b_2)$.

The *stable Grothendieck–Witt equivalence* (sGW) property: given ϵ-symmetric objects $(\mathscr{E}_1, b_1)$, $(\mathscr{E}_2, b_2)$, $(\mathscr{M}_1, h_1)$, $(\mathscr{M}_2, h_2)$, and $(\mathscr{F}, b)$ of C with $(\mathscr{M}_1, h_1)$ and $(\mathscr{M}_2, h_2)$ metabolic with isomorphic lagrangians, if $(\mathscr{E}_1, b_1) \perp (\mathscr{M}_1, h_1) \perp (\mathscr{F}, b) \cong (\mathscr{E}_2, b_2) \perp (\mathscr{M}_2, h_2) \perp (\mathscr{F}, b)$ then $(\mathscr{E}_1, b_1) \cong (\mathscr{E}_2, b_2)$. Equivalently, $(\mathscr{E}_1, b_1) \cong (\mathscr{E}_2, b_2)$ if they have the same Grothendieck–Witt class.

A scheme X is said to satisfy a property if for every line bundle $\mathscr{L}$ and every $\epsilon \in \{\pm 1\}$, the property is satisfied for ϵ-symmetric objects of the exact category $\mathsf{VB}(X)$ with the duality given by $\mathscr{H}om(-, \mathscr{L})$.

Corollary 4.6. *Let X be a scheme with 2 invertible satisfying (sM). Then a vector bundle $\mathscr{V}$ of rank 4 has a sub- or quotient line bundle if and only if $\lambda(\mathscr{V}) = 0$ in $W^4(X, \det \mathscr{V})$, equivalently, $e(\mathscr{V}) = 0$ in $W^4(X, \det \mathscr{V}^\vee)$.*

Proof. The claim concerning $\lambda(\mathscr{V})$ is a direct consequence of Corollary 4.4 and the property (M), under which the form $\lambda^2 \mathscr{V}$ is metabolic if and only if the class $\lambda(\mathscr{V}) \in W^4(X, \det \mathscr{V})$ vanishes. For the final equivalence, note that by Proposition 3.2, we have $e(\mathscr{V}) = \langle -1 \rangle \otimes \lambda(\mathscr{V}^\vee)$ in $W^4(X, \det \mathscr{V}^\vee)$. Thus $e(\mathscr{V})$ vanishes in $W^4(X, \det \mathscr{V}^\vee)$ if and only if $\lambda(\mathscr{V}^\vee)$ does (if and only if $\lambda(\mathscr{V})$ does, see Remark 3.3). $\qquad\square$

We will spend the rest of this section exploring these properties. First, note that we have the implication (sGW) $\Rightarrow$ (Wc).

Proposition 4.7. *Over a noetherian affine scheme X with 2 invertible, we have the following implications (Wc) $\Rightarrow$ (sGW) $\Rightarrow$ (sM).*

Proof. We fix, and then suppress the dependence on, some $\epsilon \in \{\pm 1\}$. Over an affine scheme with 2 invertible, metabolic is equivalent to hyperbolic; see [25, §3 Prop. 1, Cor. 1]. In particular, metabolic forms having isomorphic lagrangians are isometric. Hence (Wc) $\Rightarrow$ (sGW). Now, if $(\mathscr{E}, b)$ is stably metabolic, then there exist vector bundles $\mathscr{N}_1$ and $\mathscr{N}_2$ such that $(\mathscr{E}, b) \perp H(\mathscr{N}_1) \cong H(\mathscr{N}_2)$. Since X is affine, there exists a vector bundle $\mathscr{N}_3$ such that $\mathscr{N}_1 \oplus \mathscr{N}_3 \cong \mathscr{O}_X^n$ is free. Thus $(\mathscr{E}, b) \perp H(\mathscr{O}_X^n) \cong H(\mathscr{N})$, where $\mathscr{N} = \mathscr{N}_1 \oplus \mathscr{N}_3$. Inspired by Roy [34, Ex. 7.3], we choose a hyperbolic pair e and f of $H(\mathscr{O}_X)$ and consider the image $v + \varphi$ in $H(\mathscr{N})$ of $e + f$. Then $\varphi(v) = f(e) = 1$, so that v is a unimodular element generating a free direct summand of $\mathscr{N}$. In particular, $H(\mathscr{N}) \cong H(\mathscr{O}_X) \oplus H(\mathscr{N}')$. By (Wc), we can cancel $H(\mathscr{O}_X)$. By induction, we deduce that $(\mathscr{E}, b)$ is hyperbolic, hence (sM) holds. $\qquad\square$

There may exist stably metabolic objects that are not metabolic. Such an example due to Ojanguren (see [6, Ex. 40]): the reduced norm form associated to the endomorphism algebra of the tangent bundle of the real 2-sphere $\operatorname{Spec} \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$.

Definition 4.8. We say that an exact category C satisfies the *Krull–Schmidt property* (KS) if every object has a unique (up to permutation) coproduct decomposition into indecomposable objects, see Atiyah [1, Thm. 3]. Additionally, we say that C satisfies the *strong Krull–Schmidt property* (KS+) if for every object $\mathscr{E}$ the ring $\operatorname{End}_C(\mathscr{E})$ is complete with respect to the $\operatorname{rad}(\operatorname{End}_C(\mathscr{E}))$-adic topology, cf. [33, §3(iii)] or [26, II §6.3].

For example, (KS+) is satisfied in the following cases: the exact category $\mathsf{VB}(X)$ of vector bundles over a scheme X proper over a field (see [1, Cor. Lemma 10] and [26, II §7] or [32, §4.1]), and the category of projective modules over a (possibly noncommutative) artinian ring (see [26, II Ex. 6.6.2]).

It is known that (KS+) $\Rightarrow$ (Wc), see [32, Satz 3.4(iii)], [33, §3.4(1)], or [26, II Thm. 6.6.1]. In particular, the spectrum of an artinian ring satisfies (M) by Proposition 4.7.

Let Σ be a set of indecomposable objects of an exact category C. An object $\mathscr{E}$ is of *type* Σ (resp. Σ') if every indecomposable direct summand of $\mathscr{E}$ is isomorphic to some element of Σ (resp. if no indecomposable direct summand is isomorphic to some element of Σ). An important consequences of the strong Krull–Schmidt property is the Krull–Schmidt theorem for symmetric objects, see [33, Thm. 3.2–3.3(1)] or [26, II Thm. 6.3.1].

Theorem 4.9 (Symmetric Krull–Schmidt). *Let* $(\mathsf{C}, \sharp, \varpi)$ *be an exact category with duality. If* C *satisfies (KS) then every* ϵ-*symmetric object* $(\mathscr{E}, b)$ *has an orthogonal decomposition*

$$(\mathscr{E}, b) \cong \overset{r}{\underset{i=1}{\perp}} (\mathscr{E}_i, b_i)$$

with $\mathscr{E}_i$ *of type* $\{\mathscr{N}_i, \mathscr{N}_i^\sharp\}$ *and* $\mathscr{N}_i \oplus \mathscr{N}_i^\sharp \not\cong \mathscr{N}_j \oplus \mathscr{N}_j^\sharp$ *for* $i \neq j$. *Moreover, if* C *satisfies (KS+), then this orthogonal decomposition is unique up to isometry and permutation.*

Remark 4.10. Two metabolic objects having isomorphic lagrangians need not share Krull–Schmidt decompositions. For example, over $\mathbb{P}^1$, $\mathscr{O}(-1)^2$ can be presented as a lagrangian of the hyperbolic form $H(\mathscr{O}^2)$, hence $H(\mathscr{O}^2)$ and $H(\mathscr{O}(-1)^2)$ have isomorphic lagrangians, yet have very different Krull–Schmidt decompositions.

We recall the following generalization of a result of Grothendieck [22], cf. [33, §3.4(3)] or [26, II Prop. 7.1.1].

Theorem 4.11. *Let* k *be an algebraically closed field and* $(\mathsf{C}, \sharp, \varpi)$ *an exact* k-*category with duality satisfying (KS+). Then* ϵ-*symmetric objects* $(\mathscr{E}_1, b_1)$ *and* $(\mathscr{E}_2, b_2)$ *are isometric if and only if* $\mathscr{E}_1 \cong \mathscr{E}_2$.

If an ϵ-symmetric object $(\mathscr{E}, b)$ has type $\{\mathscr{N}, \mathscr{N}^\sharp\}$, with $\mathscr{N}$ indecomposable and $\mathscr{N} \not\cong \mathscr{N}^\sharp$, then in fact $(\mathscr{E}, b) \cong H(\mathscr{N}^r)$ is hyperbolic (where r is such that $\mathscr{E} \cong \mathscr{N}^r \oplus \mathscr{N}^{\sharp r}$), see [33, Thm. 3.3(3)] or [26, II Prop. 6.4.1]. Otherwise, $(\mathscr{E}, b)$ is $\mathscr{N}$-*isotypic* if it has type $\{\mathscr{N}\}$ with $\mathscr{N}$ indecomposable (then the $\mathscr{N}$-*rank* is well-defined).

Corollary 4.12. *Retain the hypotheses of Theorem 4.11 and assume that* k *has characteristic* $\neq 2$. *Let* $\mathscr{N}$ *be an indecomposable object and* $(\mathscr{E}, b)$ *an* $\mathscr{N}$-*isotypic* ϵ-*symmetric object. If* $\mathscr{N}$ *has an* ϵ-*symmetric structure then* $(\mathscr{E}, b) \cong H(\mathscr{N}^m)$ *or* $(\mathscr{E}, b) \cong H(\mathscr{N}^m) \perp \mathscr{N}$, *depending on whether the* $\mathscr{N}$-*rank of* $\mathscr{E}$ *is* $2m$ *or* $2m + 1$, *respectively. Furthermore, if* $\mathscr{N}$ *is self-dual but has no* ϵ-*symmetric structure, then* $(\mathscr{E}, b)$ *is hyperbolic.*

Proof. Let $\mathscr{E}$ have $\mathscr{N}$-rank n. If $\mathscr{N}$ has an ϵ-symmetric structure, then $(\mathscr{E}, b) \cong \mathscr{N}^{\perp n}$ by Theorem 4.11. Thus $(\mathscr{E}, b)$ has a symmetric summand $\mathscr{N}^{\perp 2m}$, where $n = 2m$ or $n = 2m + 1$. But $\mathscr{N}^{2m} \cong \mathscr{N}^m \oplus (\mathscr{N}^\sharp)^m$ and hence, again by Theorem 4.11, there's an isometry $\mathscr{N}^{\perp 2m} \cong H(\mathscr{N}^m)$. If $\mathscr{N}$ has no ϵ-symmetric structure, the statement follows by the "reduction theorem" technique [33, Thm. 3.3(3)]. $\square$

Finally, we show that under the hypotheses of Theorem 4.11, a proof of (sM) can be broken into two components.

Proposition 4.13. *Let* k *be an algebraically closed field of characteristic* $\neq 2$ *and* $(\mathsf{C}, \sharp, \varpi)$ *an exact* k-*category with duality satisfying (KS+). Assume that:*

(1) *If* $\mathscr{N}$ *is a self-dual indecomposable object then any* $\mathscr{N}$-*isotypic part of a metabolic object is metabolic.*

(2) *If* $\mathscr{N}$ *is an indecomposable* ϵ-*symmetric object such that* $\mathscr{N} \perp H(\mathscr{N}^m)$ *is metabolic then* $\mathscr{N}$ *is metabolic.*

Then $(\mathsf{C}, \sharp, \varpi)$ *satisfies (sM).*

Proof. Let $(\mathscr{E}, b)$ be an ϵ-symmetric stably metabolic object and write $(\mathscr{E}, b) \perp (\mathscr{M}_1, b_1) \cong (\mathscr{M}_2, b_2)$ for metabolic objects $(\mathscr{M}_1, b_1)$ and $(\mathscr{M}_2, b_2)$. If every element of the symmetric Krull–Schmidt decomposition of $(\mathscr{E}, b)$ has type $\{\mathscr{N}, \mathscr{N}^\sharp\}$ with $\mathscr{N} \not\cong \mathscr{N}^\sharp$, then already $(\mathscr{E}, b)$ is hyperbolic. Otherwise, choose an $\mathscr{N}$-isotypic part. Using assumption (1), we can reduce to the case where $(\mathscr{E}, b)$, $(\mathscr{M}_1, b_1)$, and $(\mathscr{M}_2, b_2)$ are all $\mathscr{N}$-isotypic.

First, if $\mathscr{N}$ has no structure of ϵ-symmetric object, then $(\mathscr{E}, b)$ is already hyperbolic by Corollary 4.12. Thus we may assume that $\mathscr{N}$ has a structure of ϵ-symmetric object. If $\mathscr{E}$

has even $\mathcal{N}$-rank, the by Corollary 4.12, $(\mathscr{E}, b)$ is already hyperbolic. If $\mathscr{E}$ as odd $\mathcal{N}$-rank, then either $\mathscr{M}_1$ or $\mathscr{M}_2$ has odd $\mathcal{N}$-rank. But then Corollary 4.12 together with assumption (2) implies that $\mathcal{N}$ is metabolic and thus $(\mathscr{E}, b)$ is metabolic. $\qquad\square$

Using Proposition 4.13, we can verify (sM) in some cases. For example, if $X = \mathbb{P}^1$ over an algebraically closed field of characteristic $\neq 2$, then indecomposable vector bundles are just line bundles (see [22]). For a given duality $\mathscr{H}om(-, \mathscr{L})$ on $\mathsf{VB}(X)$, there is at most one self-dual indecomposable line bundle, namely $\mathcal{N} = \mathcal{O}(d)$ if $\mathscr{L} \cong \mathcal{O}(2d)$ is even. In particular, an $\mathcal{N}$-isotypic part is hyperbolic if and only if it has even rank. Thus, both hypotheses (1) and (2) of Proposition 4.13 are verified over $\mathbb{P}^1$.

We wonder if (sM) holds for any scheme proper over a field of characteristic $\neq 2$.

4.3. **On the vanishing of the Euler class.** In this section, we investigate necessary conditions for the vanishing of the Grothendieck–Witt-theoretic Euler class $e(\mathscr{V}) \in GW^r(X, \det \mathscr{V}^\vee)$ of a vector bundle $\mathscr{V}$ of rank r on a scheme X with 2 invertible. The following important special case of Corollary 3.4 gives a sufficient condition for the vanishing of the Euler class.

Proposition 4.14 (Fasel–Srinivas [19, Prop. 22]). *Let X be a scheme with 2 invertible and $\mathscr{V}$ a vector bundle of rank r on X. If $\mathscr{V}$ has a free sub- or quotient line bundle then $e(\mathscr{V}) = 0$ in $GW^r(X, \det \mathscr{V}^\vee)$.*

One may ask when the existence of a free sub- or quotient line bundle is equivalent to the vanishing of the Grothendieck–Witt-theoretic Euler class of a vector bundle. For vector bundles of rank one, this question always has a positive answer; the following simple argument was communicated to the author by J. Fasel.

Proposition 4.15. *Let X be a scheme with 2 invertible and $\mathscr{L}$ be a line bundle on X. Then $\mathscr{L} \cong \mathcal{O}_X$ if and only if $e(\mathscr{L}) = 0$ in $GW^1(X, \det \mathscr{L}^\vee)$.*

Proof. By Proposition 3.2 part 1, we have $e(\mathscr{L}) = H_{\mathscr{L}^\vee}(\mathcal{O}_X) \in GW^1(X, \mathscr{L}^\vee)$. In particular, $e(\mathscr{L})$ maps to $[\mathcal{O}_X] - [\mathscr{L}^\vee]$ under the forgetful map $GW^1(X, \mathscr{L}^\vee) \to K_0(X)$. This, in turn, maps to the isomorphism class of $\mathscr{L}$ under the determinant homomorphism $\det : K_0(X) \to \mathrm{Pic}(X)$. Thus if $e(\mathscr{L}) = 0$ then $\mathscr{L}$ is trivial. $\qquad\square$

There exist vector bundles of rank 2 with no sub- or quotient line bundle but with vanishing Euler class: such examples arise as stably metabolic but nonmetabolic skew-symmetric forms of rank 2 and trivial determinant. However, if the stable metabolicity property (sM) is satisfied (see Definition 4.5), this obstruction should disappear.

Question 4.16. *Let X be a scheme with 2 invertible and satisfying (KS+) and (sM). Let $\mathscr{V}$ be a vector bundle of rank 2. Is the vanishing the the Euler class $e(\mathscr{V}) \in GW^2(X, \det \mathscr{V}^\vee)$ equivalent to $\mathscr{V}$ having a free sub- or quotient line bundle?*

To give some partial results, we can proceeding analogously as in the proof of Corollary 4.4, first developing the rank 2 version of Theorem 4.3 utilizing the exceptional isomorphism $\mathsf{A}_1 = \mathsf{C}_1$.

Theorem 4.17. *Let X be a scheme with 2 invertible and $\mathscr{V}$ a vector bundle of rank 2 on X. Then*
$$\Phi_{\mathscr{V}} : \mathbb{P}(\mathscr{V}) \to \Lambda\mathbb{G}(\lambda^1 \mathscr{V})$$
is an isomorphism of X-schemes.

Proof. The proof is similar to, except easier than, that of Theorem 4.3. In place of Proposition 4.1, we use the fact that the functor λ^1 defines an isomorphism of group schemes $\mathbf{GL}(\mathscr{V}) \cong \mathbf{GSp}(\lambda^1 \mathscr{V})$ over X, inducing the required isomorphism of projective homogeneous bundles. $\qquad\square$

In particular, we have the following analogue of Corollary 4.4.

Corollary 4.18. *A vector bundle $\mathscr{V}$ of rank 2 fits into a short exact sequence $0 \to \mathcal{N} \to \mathscr{V} \to \mathscr{L} \to 0$ for line bundles $\mathcal{N}$ and $\mathscr{L}$ if and only if $\lambda^1 \mathscr{V}$ is metabolic.*

Finally, under some stability hypotheses, we can answer Question 4.16 in the affirmative.

Proposition 4.19. *Let X be a scheme with 2 invertible and satisfying (sM) and (sGW). Let $\mathscr{V}$ be a vector bundle of rank 2 on X. Then $\mathscr{V}$ has a free sub- or quotient line bundle if and only if $e(\mathscr{V}) = 0$ in $GW^2(X, \det \mathscr{V}^\vee)$.*

Proof. By Proposition 3.2 part (1), we have that

$$e(\mathscr{V}) = \lambda(\mathscr{V}^\vee) + H_{\det \mathscr{V}^\vee}(\mathscr{O}_X)$$

in $GW^2(X, \det \mathscr{V}^\vee)$. If $e(\mathscr{V}) = 0$, we see that $\lambda(\mathscr{V}^\vee) \in GW^2(X, \det \mathscr{V}^\vee)$ is a metabolic class. Thus $\lambda^1 \mathscr{V}$ is metabolic by (sM). By Corollary 4.18, there is an exact sequence $0 \to \mathscr{N} \to \mathscr{V} \to \mathscr{L} \to 0$. But then by Proposition 3.5, we have $e(\mathscr{V}) = H_{\det \mathscr{V}^\vee}([\mathscr{O}_X] - [\mathscr{N}^\vee])$. Thus $H_{\det \mathscr{V}^\vee}(\mathscr{O}_X)$ and $H_{\det \mathscr{V}^\vee}(\mathscr{N}^\vee)$ have equal Grothendieck–Witt classes, hence $H_{\det \mathscr{V}^\vee}(\mathscr{O}_X) \cong H_{\det \mathscr{V}^\vee}(\mathscr{N}^\vee)$ by (sGW). The classification of binary line bundle-valued quadratic forms (cf. [2, §5.2]) in terms of norm forms associated to line bundles on étale quadratic covers of X (in this case, the split cover) shows that either $\mathscr{N}^\vee$ or $\mathscr{H}om(\mathscr{N}^\vee, \det \mathscr{V}^\vee) \cong \mathscr{L}^\vee$ is free. Thus $\mathscr{V}$ has a free sub- or quotient line bundle. $\square$

At the other extreme (i.e. when neither (sM) nor (sGW) are expected to hold), one can consider the case when X is a noetherian affine scheme (with 2 invertible) of dimension d, where vector bundles are just projective modules of finite rank. The question of whether the Euler class in Grothendieck-Witt theory is the only obstruction to "splitting off a free factor" has a positive answer when $d = 2$ (rank 2 modules being the only case not covered by Serre's splitting theorem, see Remark 2.5) by Fasel–Srinivas [19, Thm. 3], who also handle the case of projective modules of rank ≥ 3 when $d = 3$. The case $d = 3$ and projective modules of rank 2 with trivial determinant follows by the discussion surrounding Fasel [17, Thm. 4.3]. In general, one can only hope that the Euler class is the obstruction to splitting off a free factor for projective modules of rank $\geq d$ (rank d being the crucial case), a question asked in [19].

As for vector bundles of rank four, we have the following partial result under a stability hypothesis.

Proposition 4.20. *Let X be a scheme with 2 invertible and satisfying (sGW). Let $\mathscr{V}$ be a vector bundle of rank 4 with $\det \mathscr{V} \cong \mathscr{O}_X$. Then $\mathscr{V}$ has a sub- or quotient line bundle if and only if $e(\mathscr{V}) = 0$ in $GW^4(X)$.*

Proof. By Proposition 3.2, we have

$$e(\mathscr{V}) = \langle -1 \rangle \otimes \lambda(\mathscr{V}^\vee) + H\big(\mathscr{O}_X - \mathscr{V}^\vee\big).$$

Supposing that $e(\mathscr{V}) = 0$, the forms $\lambda^2 \mathscr{V}^\vee \perp H(\mathscr{O}_X)$ and $H(\mathscr{V})$ become equivalent in $GW^4(X)$. Hence

$$\lambda^2 \mathscr{V}^\vee \perp H(\mathscr{O}_X) \cong H(\mathscr{V})$$

by (sGW). Inspired by Roy [34, Ex. 7.3], we now proceed similarly as in the proof of Proposition 4.7. Choose a hyperbolic basis of global sections e and f of $H(\mathscr{O}_X)$ and consider the image $v + \varphi$ in $H(\mathscr{V})$ of $e + f$. Then $\varphi(v) = f(e) = 1$, so that v is a unimodular element generating a free factor of $\mathscr{V}$. $\square$

We wonder what the optimal result is in the case of vector bundles $\mathscr{V}$ of rank 4. One approach, suggested by the referee, works over smooth affine schemes X of dimension 4 over a field k of characteristic $\neq 2$. The idea is to use the Chow–Witt-theoretic top Chern class $\tilde{c}_4(\mathscr{V}) \in \widetilde{CH}^4(X, \det \mathscr{V}^\vee)$, which is the precise obstruction to $\mathscr{V}$ having a free factor by a theorem of Morel [29, Thm. 7.14]. See [15] for more details on the top Chern class in Chow–Witt theory. The Gersten–Grothendieck–Witt spectral sequence provides an edge homomorphism $\widetilde{CH}^4(X, \det \mathscr{V}^\vee) \to GW^4(X, \det \mathscr{V}^\vee)$ mapping $\tilde{c}_4(\mathscr{V})$ to $e(\mathscr{V})$, see Fasel–Srinivas [19, §4.2]. The interest then lies in the kernel of this edge homomorphism.

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