MAXIMAL BRILL–NOETHER LOCI VIA K3 SURFACES

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Abstract. The Brill–Noether loci $\mathcal{M}^r_{g,d}$ parameterize curves of genus g admitting a linear system of rank r and degree d; when the Brill–Noether number is negative, they sit as proper subvarieties of the moduli space of genus g curves. We explain a strategy for distinguishing Brill–Noether loci by studying the lifting of linear systems on curves in polarized K3 surfaces, which motivates a conjecture identifying the maximal Brill–Noether loci. Via an analysis of the stability of Lazarsfeld–Mukai bundles, we obtain new lifting results for line bundles of type g^3_d which suffice to prove the maximal Brill–Noether loci conjecture in genus 9–19, 22, and 23.

Introduction

Given a smooth projective complex curve C of genus g, classical Brill–Noether theory concerns the geometry of the variety $W^r_d(C)$, parameterizing the space of line bundles of type g^r_d, i.e., having degree d and at least $r+1$ linearly independent global sections on C. Specifically, the expected dimension of $W^r_d(C)$ is the *Brill–Noether number* $\rho(g,r,d) := g-(r+1)(g-d+r)$. In particular, when $\rho(g,r,d)\geq 0$, every smooth curve of genus g admits a line bundle of type g^r_d. If $\rho(g,r,d)<0$, then a curve admitting such a g^r_d is called Brill–Noether special, and the *Brill–Noether locus* $\mathcal{M}^r_{g,d}$ parametrizing smooth curves of genus g admitting a line bundle of type g^r_d is a proper subvariety of the moduli space $\mathcal{M}_g$ of smooth curves of genus g, see [1].

In general, the geometry of Brill–Noether loci is complicated by the existence of multiple components with some that are non-reduced or not of the expected dimension. Indeed, while the Brill–Noether locus $\mathcal{M}^r_{g,d}$ has expected codimension $-\rho$ in $\mathcal{M}_g$, the actual codimension of its components is bounded above by $-\rho$ when $\rho<0$, see e.g., [13], but it could be lower, and known examples with lower than expected codimension exist when $-\rho>g-3$, see [35]. On the other hand, when $\rho(g,r,d)=-1$, Eisenbud and Harris [10] show that $\mathcal{M}^r_{g,d}$ is irreducible of codimension 1. More generally, when $-3\leq\rho\leq -1$, any component of $\mathcal{M}^r_{g,d}$ has codimension $-\rho$, see [8, 10, 38]. The Brill–Noether divisors were used by Harris, Mumford, and Eisenbud [9, 17, 16] in their investigation of the Kodaira dimension of $\mathcal{M}_g$ when $g\geq 23$.

A question of interest is then to determine the stratification of $\mathcal{M}_g$ by Brill–Noether loci and, in particular, to identify those loci that are maximal with respect to containment. For Brill–Noether divisors, this is equivalent to having distinct support, a property that is crucially used by Eisenbud and Harris [9], and further developed by Farkas [11], to give lower bounds on the Kodaira dimension of $\mathcal{M}_{23}$. There are various trivial containments among the Brill–Noether loci, e.g., $\mathcal{M}^1_{g,2}\subseteq\mathcal{M}^1_{g,3}\subseteq\cdots\subseteq\mathcal{M}^1_{g,k}=\mathcal{M}_g$, where $k\geq\lfloor\frac{g+3}{2}\rfloor$ is at least the generic gonality of a curve of genus g. Likewise, we have $\mathcal{M}^r_{g,d}\subseteq\mathcal{M}^r_{g,d+1}$ by adding a base point to a g^r_d on C. Similarly, by subtracting a point not in the base locus, $\mathcal{M}^r_{g,d}\subseteq\mathcal{M}^{r-1}_{g,d-1}$ when $\rho(g,r-1,d-1)<0$, see [12, 25]. Modulo these trivial containments, the *expected maximal Brill–Noether loci* are the $\mathcal{M}^r_{g,d}$, where for fixed r, d is maximal such that $\rho(g,r,d)<0$ and $\rho(g,r-1,d-1)\geq 0$. In this work, we conjecture that the expected maximal Brill–Noether loci are indeed maximal, and we verify the conjecture in certain genera by developing new results on lifting linear systems on curve in K3 surfaces along with a general program of relating such lifting results to the containment of Brill–Noether loci.

Conjecture 1. In genus $g\geq 9$, the maximal Brill–Noether loci are the expected ones. That is, at least one component of each expected maximal Brill–Noether locus is not contained in any other Brill–Noether locus of $\mathcal{M}_g$.

Hence, we are interested in distinguishing the expected maximal Brill–Noether loci. In low genus, there are nontrivial containments among the expected maximal Brill–Noether loci. In fact, in genus ≤ 8 there is a unique maximal Brill–Noether locus, despite the fact that in genus 7 and 8 there are two expected maximal Brill–Noether loci. For example, Mukai [31] proved that every Brill–Noether special curve of genus 8 has a g^2_7, cf. [2, Lemma 1.2]. Recently, there have been several breakthroughs in the study of Brill–Noether special curves of fixed gonality [6, 19, 23, 24, 33, 34], from which one can deduce that the expected maximal $\mathcal{M}^1_{g,\lfloor\frac{g+1}{2}\rfloor}$ is not contained in any of the other expected maximal loci and hence is maximal, see Section 1. Additionally, there has been recent focus on showing that Brill–Noether loci of codimension 1 and 2 are distinct, and showing various non-containments of Brill–Noether loci of codimension 2, see [3, 4, 5, 21]; in fact, for $g \geq 34$ and not divisible by 3, one can deduce that there are at least 2 maximal Brill–Noether loci. These results are proved using a mix of tropical, combinatorial, and limit linear series methods.

On the other hand, our approach is to use K3 surfaces to construct curves admitting a g^r_d, but not a $g^{r'}_{d'}$, thus distinguishing the Brill–Noether loci. This idea was introduced by Farkas [12], and further developed by Lelli-Chiesa [26, 28], who can produce curves on a K3 surface admitting a g^1_d or g^2_d, but not a $g^r_{d'}$. We further extend this technique to curves that admit a g^3_d, which suffices to prove our main theorem.

Theorem 1. Conjecture 1 holds in genus 9–19, 22, and 23.

Concerning genus 20 and 21, our results reduce Conjecture 1 to the verification that the codimension of $\mathcal{M}^3_{20,17}$ and $\mathcal{M}^4_{21,20}$ is the expected value of 4, and that the codimension of $\mathcal{M}^4_{20,19}$ is at least the expected value of 5.

The geometry of polarized K3 surfaces is intimately related to the Brill–Noether theory of curves C in the polarization class, see e.g., [20, 22, 29, 30, 36, 37]. Foundational to this is Green and Lazarsfeld's celebrated result that the Clifford index $\gamma(C)$ is constant as C moves in its linear system [14]. Donagi and Morrison [7, Theorem 5.1'] proved that if A is a complete basepoint free Brill–Noether special g^1_d on a non-hyperelliptic smooth curve $C \in |H|$, then $|A|$ is contained in the restriction of $|M|$ for a line bundle $M \in \mathrm{Pic}(S)$.

In fact, they conjectured that this is always true, with some slight modifications due to Lelli-Chiesa.

Conjecture 2 (Donagi–Morrison Conjecture, [27] Conjecture 1.3)**.** Let (S, H) be a polarized K3 surface and $C \in |H|$ be a smooth irreducible curve of genus ≥ 2. Suppose A is a complete basepoint free g^r_d on C such that $d \leq g-1$ and $\rho(g, r, d) < 0$. Then there exists a line bundle $M \in \mathrm{Pic}(S)$ adapted to $|H|$ such that $|A|$ is contained in the restriction of $|M|$ to C and $\gamma(M \otimes \mathcal{O}_C) \leq \gamma(A)$.

For further details and definitions, see Section 1.1. Lelli-Chiesa has verified the Donagi–Morrison conjecture for linear systems of type g^2_d under some mild hypotheses [26], and more recently [27] has proven the conjecture if the pair (C, A) does not have unexpected secant varieties up to deformation. The proofs of these results use Lazarsfeld–Mukai bundles $E_{C,A}$ associated to the pair (C, A), and the fact that when the vector bundle $E_{C,A}$ has a nontrivial maximal destabalizing sub-line bundle $N \in \mathrm{Pic}(S)$, then $|A|$ is contained in the restriction of $|H \otimes N^\vee|$. For rank 2 linear systems, a case-by-case analysis of the Jordan–Hölder and Harder–Narasimhan filtrations of $E_{C,A}$ is used. This technique becomes much more difficult in higher rank. In general, Lelli-Chiesa [27, Theorem 4.2] proves that A does lift when it computes the Clifford index $\gamma(C)$. However, in genus $g \geq 14$, except for $\mathcal{M}^1_{g,\lfloor\frac{g+1}{2}\rfloor}$, all of the expected maximal Brill–Noether loci correspond to *non-computing* Brill–Noether special linear systems, i.e., linear systems $|A|$ with $\rho(A) < 0$ and $\gamma(A) > \lfloor\frac{g-1}{2}\rfloor$ so that A cannot compute the Clifford index, as $\gamma(C) \leq \lfloor\frac{g-1}{2}\rfloor$.

Our main lifting result is a proof of the Donagi–Morrison conjecture for linear systems of rank 3 and bounded degree.

Theorem 2. Let (S, H) be a polarized K3 surface of genus $g \neq 2, 3, 4, 8$ and $C \in |H|$ a smooth irreducible curve of Clifford index $\gamma(C)$. Suppose that S has no elliptic curves and $d < \frac{5}{4}\gamma(C) + 6$, then Conjecture 2 holds for any g^3_d on C. Moreover, one has $c_1(M).C \leq \frac{3g-3}{2}$.

We prove a slightly more refined version, replacing the hypothesis on non-existence of elliptic curves with an explicit dependence on the Picard lattice of S, see Theorem 5.1.

With this lifting result in hand, Theorem 1 is proved by considering K3 surfaces (S, H) with a prescribed Picard group so that curves $C \in |H|$ have a g^r_d, and then proving that if C had a $g^3_{d'}$, its Donagi–Morrison lift would not be compatible with the Picard group. This latter argument involves some elementary lattice theory. More generally, we explain how a Donagi–Morrison type result together with some lattice theory imply Conjecture 1. As the Donagi–Morrison conjecture is not known in rank 4 and above, we cannot show that some of the expected maximal Brill–Noether loci are not contained in the $\mathcal{M}^4_{g,d}$ in genus 20, 21, and ≥ 24. In genus 22 and 23, known results about the codimension of components of Brill–Noether loci and non-containments of codimension 2 loci, together with our results, suffice to distinguish the expected maximal loci.

Outline. In Section 1, we briefly analyze some constraints on lifting line bundles and find that in genus ≥ 14 the expected maximal Brill–Noether loci correspond to line bundles that cannot compute the Clifford index of the curve, and summarize how Conjecture 2 implies Conjecture 1. The following two sections, Section 2 and Section 3, provide some background on the notion of stability of coherent sheaves on K3 surfaces and on Lazarsfeld–Mukai bundles and their relation to lifting line bundles. We also briefly recall some useful facts about generalized Lazarsfeld–Mukai bundles which are needed in particular arguments. At the end of Section 3, we motivate our proof strategy in Proposition 3.14. In Section 4, we first reduce the problem to finding a bound for each terminal filtration of the Lazarsfeld–Mukai bundle associated to the g^3_d, a filtration obtained by taking the Harder–Narasimhan and Jordan–Hölder filtrations of the Lazarsfeld–Mukai bundle. We then find a bound on the degree of the g^3_d for each filtration. In Section 5, after having obtained bounds for every terminal filtration that does not have a maximal destabilizing sub-line bundle, we give the proof of Theorem 2. Finally, in Section 6, we use known results about dimensions of components of Brill–Noether loci and other lifting results to prove Theorem 1. In Section 7, we prove the results in genus $9 - 13$.

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1. Maximal Brill–Noether Loci

In this section, we take a look at the analytic geometry of various Brill–Noether theory conditions on linear systems. We find simple bounds on the maximal Clifford index of Brill–Noether special linear systems and for linear systems that can potentially lift to a K3 surface without contradicting the Hodge index theorem. Furthermore, we find that all non-computing linear systems are always potentially liftable to K3 surfaces. We end with a discussion of how Conjecture 2 and lattice theory can imply Conjecture 1. We work with a fixed genus g throughout this section.

Let (S, H) be a polarized K3 surface of genus g. In the moduli space $\mathcal{K}^\circ_g$ of polarized K3 surfaces of genus g, the Noether–Lefschetz (NL) locus parameterizes K3 surfaces with Picard rank > 1. By Hodge theory, the NL locus is a union of countably many irreducible divisors, which we call NL divisors. In [15], Greer, Li, and Tian study the Picard group of $\mathcal{K}_g^\circ$ using Noether–Lefschetz theory and the locus of Brill–Noether special K3 surfaces in $\mathcal{K}_g^\circ$ is identified as a union of NL divisors. More generally, it is convenient to work with the moduli space of primitively quasi-polarized K3 surfaces, denoted $\mathcal{K}_g$ where $\mathcal{K}_g \setminus \mathcal{K}_g^\circ$ is a divisor parameterizing K3 surfaces containing a (-2)-exceptional curve. We define the NL divisor $\mathcal{K}_{g,d}^r$ to be the locus of polarized K3 surfaces $(S, H) \in \mathcal{K}_g$ such that

$$\Lambda_{g,d}^r = \begin{array}{c|cc} & H & L \\ \hline H & 2g-2 & d \\ L & d & 2r-2 \end{array}$$

admits a primitive embedding in $\mathrm{Pic}(S)$ preserving H. We note that the $\mathcal{K}_{g,d}^r$ are each irreducible by [32]. As we'll show in Lemma 6.2, polarized K3 surfaces $(S, H) \in \mathcal{K}_{g,d}^r$ should be thought of as those having a curve $C \in |H|$ such that $L \otimes \mathcal{O}_C$ is a line bundle of type g_d^r, and we say that the lattice $\Lambda_{g,d}^r$ is *associated* to g_d^r. Specifically, we have the following lemma, which we prove in Section 6.

Lemma 1.1 (See Lemma 6.2)**.** *Let $(S, H) \in \mathcal{K}_{g,d}^r$ and let $C \in |H|$ be a smooth irreducible curve. If L and $H - L$ are basepoint free, $r \geq 2$, and $1 \leq d \leq g-1$, then $L \otimes \mathcal{O}_C$ is a g_d^r.*

Conversely, one is interested in the question of when a given g_d^r on a curve in a K3 surface is the restriction of a line bundle from the K3; in this case, we say that the line bundle is a *lift* of the g_d^r. Lifting of line bundles on curves on K3 surfaces is considered in [7, 14, 26, 27, 29, 36]. In lifting Brill–Noether special linear systems on $C \in |H|$ to a line bundle $L \in \mathrm{Pic}(S)$, we are naturally led to considering two constraints. First, we have $\rho(g, r, d) < 0$ as the linear system is special. We call the constraint $\rho(g, r, d) < 0$ the *Brill–Noether constraint*. If a g_d^r on a curve $C \in |H|$ on a polarized K3 surface (S, H) has a suitable lift (see Corollary 3.11), then $\mathrm{Pic}(S)$ admits a primitive embedding of $\Lambda_{g,d}^r$ preserving H, and in particular $\mathrm{disc}(\Lambda_{g,d}^r) < 0$ by the Hodge index theorem. Thus we define

$$\Delta(g, r, d) := \mathrm{disc}\left(\Lambda_{g,d}^r\right) = 4(g-1)(r-1) - d^2 = 4(g-1)(r-1) - (\gamma(r, d) + 2r)^2.$$

We thus call the constraint $\Delta(g, r, d) < 0$ the *Hodge constraint* as the inequality stems from the Hodge index theorem. We remark that when $\Delta(g, r, d) < 0$, the Torelli theorem for polarized K3 surfaces implies that a very general K3 surface in $\mathcal{K}_{g,d}^r$ has $\mathrm{Pic}(S) = \Lambda_{g,d}^r$.

Remark 1.2. When considering the lifting of linear systems to K3 surfaces, it is more convenient to consider the Brill–Noether and Hodge constraints for fixed g in the (r, γ)-plane as opposed to the (r, d)-plane, in particular, because the Clifford index of curves on K3 surfaces remains constant in their linear system [14]. In the (r, γ)-plane the Brill–Noether and Hodge constraints determine regions that are bounded by the curves $\rho(g, r, d) = 0$ and $\Delta(g, r, d) = 0$, which we call the *Brill–Noether hyperbola* and *Hodge parabola*, respectively. Simple calculations show that the maximum γ on the Brill–Noether hyperbola is obtained at $r = \sqrt{g} - 1$ and $\gamma = g - 2\sqrt{g} + 1$, the intersection with the line $d = g - 1$. Hence, taking $\gamma \leq \lfloor g - 2\sqrt{g} + 1 \rfloor$ suffices to bound Brill–Noether special linear systems. Similarly, the maximum γ on the Hodge parabola is given by $\gamma = \frac{g-5}{2}$, and obtained at the intersection with the line $d = g - 1$ at $r = \frac{g+3}{4}$. Thus if $\gamma > \frac{g-5}{2}$ then $\Delta < 0$. Trivially $\lfloor \frac{g-4}{2} \rfloor \geq \frac{g-5}{2}$, and in fact the bound $\gamma \geq \lfloor \frac{g-4}{2} \rfloor \implies \Delta < 0$ is the best possible as seen in genus 9, 13, and 17. As an example, we show the bounds in genus 100, as graphed on the (r, γ)-plane in Figure 1.

We recall that the *Clifford index* of a line bundle A on a smooth projective curve C is the integer $\gamma(A) = \deg(A) - 2\,r(A)$ where $r(A) = h^0(C, A) - 1$ is the rank of A. The Clifford index of C is

$$\gamma(C) := \min\{\gamma(A) \mid h^0(C, A) \geq 2 \text{ and } h^1(C, A) \geq 2\}.$$

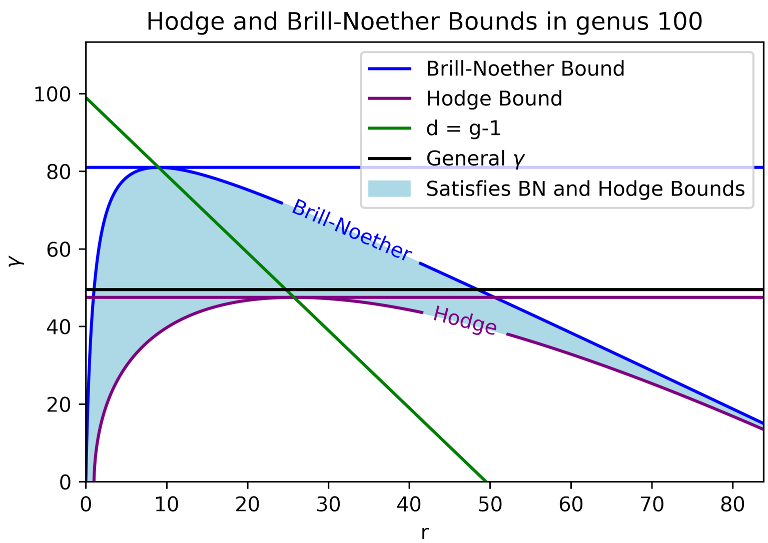

FIGURE 1. The Brill–Noether hyperbola ($\rho = 0$) and the Hodge parabola ($\Delta = 0$) in genus 100. The shaded area satisfies both $\rho < 0$ and $\Delta < 0$.

We say that a line bundle A on C *computes* the Clifford index of C if $\gamma(A) = \gamma(C)$. Clifford's theorem states that $0 \leq \gamma(C) \leq \lfloor \frac{g-1}{2} \rfloor$, and when C is a general curve of genus g, $\gamma(C) = \lfloor \frac{g-1}{2} \rfloor$.

Definition 1.3. Let A be a Brill–Noether special g^r_d on a curve C of genus g, i.e. $\rho(g, r, d) < 0$. We say A is *non-computing* if $\gamma(r, d) > \lfloor \frac{g-1}{2} \rfloor$; that is, a Brill–Noether special g^r_d that cannot compute $\gamma(C)$.

When $g < 14$, there are no non-computing g^r_ds. However, for genus $g \geq 14$, except for $\mathcal{M}^1_{g,\lfloor \frac{g+1}{2} \rfloor}$, all the maximal Brill–Noether loci are those associated to non-computing g^r_ds. If lifting results are able to distinguish between maximal Brill–Noether loci, there should not be an obvious obstruction to lifting the associated linear systems. In particular, the Hodge index theorem implies that the lattices obtained by lifting should have negative discriminant, which we show is true for non-computing g^r_ds below.

Proposition 1.4. *Let g, r, d be natural numbers with $2 \leq d \leq g - 1$ and $1 \leq r \leq g - 1$. Then the Hodge parabola lies under the Brill–Noether hyperbola. In particular, all non-computing linear systems, and all expected maximal Brill–Noether loci, satisfy $\Delta < 0$.*

Proof. For fixed $g \geq 2$, and for each constraint ($\rho = 0$ or $\Delta = 0$), we solve for γ as a function of r and g. For $\rho(g, r, \gamma) = 0$, we find $\gamma_\rho(r) = g - r - \frac{g}{r+1}$. Likewise for $\Delta(g, r, \gamma) = 0$ we have $\gamma_\Delta(r) = 2\sqrt{(g-1)(r-1)} - 2r$. Observe that $\gamma_\rho = \gamma_\Delta$ has no solutions in the given range (solve for r in terms of g, and note that $g \geq 2$). Finally, since $\gamma_\rho(1) > 0$ and $\gamma_\Delta(1) < 0$, we see by continuity that $\gamma_\rho(r) - \gamma_\Delta(r) > 0$.

The bound $\gamma \geq \lfloor \frac{g-4}{2} \rfloor$ implies that $\Delta < 0$, as in the remark above. Since this is below the general Clifford index ($\lfloor \frac{g-1}{2} \rfloor$), we see that any lattice associated to a non-computing linear system will have negative discriminant. In particular, this applies to the expected maximal linear systems. ☐

We thus conjecture (Conjecture 1) that the maximal Brill–Noether loci are exactly the *expected maximal Brill–Noether loci*, which are Brill–Noether loci $\mathcal{M}^r_{g,d}$ where for fixed r, d is maximal such that $\rho(g, r, d) < 0$ and $\rho(g, r-1, d-1) \geq 0$. Equivalently, the expected maximal Brill–Noether loci correspond to the maximal g^r_d lying under the Brill–Noether hyperbola for each r, up to the containments $\mathcal{M}^r_{g,d} \subseteq \mathcal{M}^r_{g,d+1}$ when $\rho(g,r,d+1) < 0$ and $\mathcal{M}^r_{g,d} \subset \mathcal{M}^{r-1}_{g,d-1}$ when $\rho(g,r-1,d-1) < 0$.

One could imagine that if there are any unexpected containments among Brill–Noether loci, then some would come from containments of the form $\mathcal{M}^1_{g,d} \subset \mathcal{M}^r_{g,d'}$. However, we find that the expected maximal $\mathcal{M}^1_{g,d}$ is not contained in the other expected maximal loci.

Proposition 1.5. *Let $\rho(g,r,d) < 0$, and $\gamma(r,d) \geq \lfloor \frac{g-1}{2} \rfloor + 1$, e.g., for a non-computing g^r_d. Then $\mathcal{M}^1_{g,\lfloor \frac{g+1}{2} \rfloor} \not\subseteq \mathcal{M}^r_{g,d}$.*

Proof. Let $k = \frac{g+1}{2}$, and $r' = \min\{r, g-d+r-1\}$. We compute

$$\begin{aligned}\rho_k &= \max_{\ell \in \{0,\ldots,r'\}} \rho(g, r-\ell, d) - \ell k \\ &= \max_{\ell \in \{0,\ldots,r'\}} \rho(g,r,d) + \ell \left(\left\lceil \frac{g-1}{2} \right\rceil - \gamma(r,d) + 1 \right) - \ell^2 \\ &= \max_{\ell \in \{0,\ldots,r'\}} \rho(g,r,d) + \ell - \ell^2 \leq \rho(g,r,d) < 0.\end{aligned}$$

From [34, Theorem 1.1], as $\dim W^r_d(C) \leq \rho_k$, and $W^r_d(C)$ is empty if its dimension is negative, we see that a general k-gonal curve does not admit a g^r_d. Hence $\mathcal{M}^1_{g,\lfloor \frac{g+1}{2} \rfloor} \not\subseteq \mathcal{M}^r_{g,d}$. □

In Lemma 6.2, we show that under mild assumptions, the curves $C \in |H|$ on a polarized K3 surface (S,H) with $\text{Pic}(S) = \Lambda^r_{g,d}$ associated to an expected maximal locus with $r \geq 2$ have general Clifford index. Thus the $\mathcal{M}^1_{g,\lfloor \frac{g+1}{2} \rfloor}$ does not contain other expected maximal loci in many genera.

A natural question is whether lattices corresponding to g^r_ds can be contained as sublattices in each other. In general, the answer is yes. Already in genus 14, we see that $\Lambda^2_{14,10}$ could be embedded as a sublattice of $\Lambda^2_{14,8}$. However, these are not associated to expected maximal loci. In particular, we would like to show that lattices associated to expected maximal loci cannot contain any lattices associated to other g^r_d. This turns out to be false (see Section 1.1). However, we can prove that lattices associated to Brill–Noether special linear systems with lower than general Clifford index cannot be contain in lattices associated to expected maximal loci, and that any containments between lattices associated to an expected maximal loci and those associated to non-computing g^r_ds must be equalities.

Proposition 1.6. *Let $\Lambda^r_{g,d}$ be associated to an expected maximal g^r_d.*

(i) Any lattice $\Lambda^{r'}_{g,d'}$ associated to a special $g^{r'}_{d'}$ with $\gamma(g^{r'}_{d'}) < \lfloor \frac{g-1}{2} \rfloor$ for any r' or $\gamma(g^{r'}_{d'}) = \lfloor \frac{g-1}{2} \rfloor$ if $r' \neq 1$ cannot be contained in $\Lambda^r_{g,d}$.

(ii) Let $d' \leq g-1$. Any lattice $\Lambda^{r'}_{g,d'}$ associated to another expected maximal $g^{r'}_{d'}$ is not contained in $\Lambda^r_{g,d}$, unless the lattices are isomorphic. Similarly, any lattice associated to a non-computing $g^{r'}_{d'}$ with $d' \leq g-1$ is not contained in the lattice associated to an expected maximal g^r_d unless they are isomorphic.

Proof. To simplify notation, we write Δ for the discriminant of a lattice Λ.

To prove (i), we recall that if $\Lambda_{sub} \subset \Lambda_{exp}$ is a finite index sublattice, then we have $\Delta_{sub} = [\Lambda_{exp} : \Lambda_{sub}]^2 \Delta_{exp}$. We calculate that the ratio $\frac{\Delta_{sub}}{\Delta_{exp}}$ is never a square for the lattices considered. Specifically, we show that the largest negative discriminant $-\Delta_{sub}$ among lattices with $\gamma < \lfloor \frac{g-1}{2} \rfloor$, divided by the negative discriminant $-\Delta_{exp}$ of any lattice associated to an expected maximal linear system, is not an integer. Because $\Delta(g,r,d) = \text{disc}(-\Lambda^r_{g,d}) = d^2 - 4(g-1)(r-1)$, it is clear that for fixed γ this decreases as r increases until $d = g-1$. It follows that none of the lattices considered can be contained in $\Lambda^1_{g,\lfloor\frac{g+1}{2}\rfloor}$, the expected maximal loci with $r = 1$. From now on, we assume $r > 1$. Furthermore, we can take

- $\max(-\Delta_{sub}) = d^2$ with $d = \frac{g+1}{2}$ when $\gamma = \frac{g-1}{2} - 1$; or
- $\max(-\Delta_{sub}) = d^2 - 4(g-1)$ with $d = \frac{2}{3}g + 2$ when $\gamma = \frac{g-1}{2}$.

We also note that $-\Delta$ increases when r and γ both increase by 1, and increases as γ increases for fixed r. Thus if $r' \geq r$, then clearly $\frac{\max(-\Delta_{sub})}{-\Delta_{exp}} < 1$. If $r' < r$, then moving from $g^{r'}_{d'}$ to g^r_d, we take steps increasing r' and γ by 1 until we hit r (and then take steps increasing γ) or hit the line $d = g - 1$ and we take steps increasing γ by 1 and decreasing r' by 1. Since each of these steps increase $-\Delta$, we again see that $\frac{\max(-\Delta_{sub})}{-\Delta_{exp}} < 1$. We can always take these steps since we may assume we start at $r = 1$ or $r = 2$, and the expected maximal g^r_d lie far above. Thus (i) is proved.

To prove (ii), we similarly bound $\max(-\Delta)$ and $\min(-\Delta)$ for non-computing g^r_ds. It can be verified that the ratio $\frac{\min(-\Delta)}{\max(-\Delta)} > \frac{1}{4}$ for $r < \sqrt{g}$, and hence $\max(-\Delta) < 4\min(-\Delta)$, thus the discriminants of lattices associated to the expected maximal Brill–Noether loci cannot differ by a square greater than 1. Hence if the lattices associated to expected maximal loci are contained, they must be the same lattice. Since $-\Delta$ increases as r decreases and as γ increases until $d = g - 1$, this argument in fact shows that any lattice associated to a non-expected maximal non-computing $g^{r'}_{d'}$ cannot be contained in the lattice of an expected maximal g^r_d unless they have the same discriminant. $\square$

Remark 1.7. In fact, computation up to large genus shows that the lattices associated to expected maximal loci do not contain any lattices associated to other expected maximal loci. We conjecture that this is always true, though a proof of this is currently unknown.

1.1. **Program: Lifting Results Distinguish Brill–Noether Loci.** To verify Conjecture 1, our strategy is for fixed genus g and distinct expected maximal $\mathcal{M}^r_{g,d}$ and $\mathcal{M}^{r'}_{g,d'}$ to prove that for a very general K3 surface $(S, H) \in \mathcal{K}^r_{g,d}$, a smooth curve $C \in |H|$ admits a g^r_d but not a $g^{r'}_{d'}$. We do this by combining three kinds of results: (i) a lifting result, (ii) showing that $C \in |H|$ has a g^r_d given by restricting $L \in \Lambda^r_{g,d}$, and (iii) a comparison result that distinguishes lattices. The latter two can be checked for any fixed genus. If all the lattices can be distinguished, a lifting result like the Donagi–Morrison conjecture (Conjecture 2) implies Conjecture 1.

We start by defining a few terms in Conjecture 2.

Definition 1.8. Let S be a K3 surface, $C \subset S$ be a curve, and $A \in \mathrm{Pic}(C)$ and $M \in \mathrm{Pic}(S)$ be line bundles. We say that the linear system $|A|$ is contained in the restriction of $|M|$ to C when for every $D_0 \in |A|$, there is some divisor $M_0 \in |M|$ such that $D_0 \subset C \cap M_0$.

Definition 1.9. A line bundle M is *adapted* to $|H|$ when

(i) $h^0(S, M) \geq 2$ and $h^0(S, H \otimes M^\vee) \geq 2$; and
(ii) $h^0(S, M \otimes \mathcal{O}_C)$ is independent of the smooth curve $C \in |H|$.

Thus whenever M is adapted to $|H|$, condition (i) ensures that $M \otimes \mathcal{O}_C$ contributes to $\gamma(C)$, and condition (ii) ensures that $\gamma(M \otimes \mathcal{O}_C)$ is constant as C varies in its linear system and is satisfied if either $h^1(S, M) = 0$ or $h^1(S, H \otimes M^\vee) = 0$.

Definition 1.10. Let (S, H) be a polarized K3 surface and $C \in |H|$ be a smooth irreducible curve of genus ≥ 2. Suppose A is a complete basepoint free g^r_d on C such that $d \leq g - 1$ and $\rho(g, r, d) < 0$. We call a line bundle M a *Donagi–Morrison lift* of A if M satisfies the conditions in Conjecture 2. That is,

- M is adapted to $|H|$,
- $|A|$ is contained in the restriction of $|M|$ to C, and

- $\gamma(M \otimes \mathcal{O}_C) \leq \gamma(A)$.

We call a line bundle M a *potential Donagi–Morrison lift* of A if M satisfies $\gamma(M \otimes \mathcal{O}_C) \leq \gamma(A)$ and $d(M \otimes \mathcal{O}_C) \geq d(A)$. Note that a Donagi–Morrison lift is a potential Donagi–Morrison lift. We say a (potential) Donagi–Morrison lift is of type g^s_e if $M^2 = 2s-2$ and $M.H = e$.

We summarize a few potential results distinguishing lattices, each of which would be useful in verifying Conjecture 1 given an appropriate lifting result.

(L1) For a fixed lattice $\Lambda^r_{g,d}$ associated to an expected maximal $\mathcal{M}^r_{g,d}$ and any lattice $\Lambda^{r'}_{g,d'}$ associated to another expected maximal $\mathcal{M}^{r'}_{g,d'}$, one has $\Lambda^{r'}_{g,d'} \not\subseteq \Lambda^r_{g,d}$.

(L2) For a fixed lattice $\Lambda^r_{g,d}$ associated to an expected maximal $\mathcal{M}^r_{g,d}$ and any lattice $\Lambda^{r'}_{g,d'}$ with $\lfloor \frac{g+1}{2} \rfloor \leq \gamma(r', d') \leq \lfloor g - 2\sqrt{g} + 1 \rfloor$ and $1 \leq r' \leq \lfloor \frac{g-1-\gamma(r',d')}{2} \rfloor$, one has $\Lambda^{r'}_{g,d'} \not\subseteq \Lambda^r_{g,d}$.

(L3) For a pair of lattices $(\Lambda^r_{g,d}, \Lambda^{r'}_{g,d'})$ both associated to expected maximal Brill–Noether loci, and any lattice $\Lambda^s_{g,e}$ such that $\lfloor \frac{g+1}{2} \rfloor \leq \gamma(s,e) \leq \gamma(r', d')$ and $1 \leq s \leq \lfloor \frac{g-1-\gamma(s,e)}{2} \rfloor$, one has $\Lambda^s_{g,e} \not\subseteq \Lambda^r_{g,d}$.

We note that L2 implies L1. Furthermore, for fixed r and d, L2 implies L3 for all r' and d'.

Remark 1.11. The bounds on $\gamma(s,e)$ and s in L3 include all lattices associated to a potential Donagi–Morrison lift of a $g^{r'}_{d'}$. Indeed, suppose M is a potential Donagi–Morrison lift of a $g^{r'}_{d'}$, and say M is of type g^s_e. The lower bound on $\gamma(s,e)$ comes from Proposition 1.6 (i). Since M is a potential Donagi–Morrison lift of a $g^{r'}_{d'}$, we have $\gamma(s,e) \leq \gamma(r', d')$, which is the upper bound on $\gamma(s,e)$. Since $M \otimes \mathcal{O}_C$ contributes to $\gamma(C)$, this forces $H \otimes M^\vee \otimes \mathcal{O}_C$ to be at least a g^1_{2g-2-e}, whereby $s \leq \frac{g-1-\gamma(s,e)}{2}$ as $2s \leq e$, which gives the upper bound on s.

Similarly, the bounds in L2 include all lattices associated to a potential Donagi–Morrison lift of an expected maximal linear system. $M \otimes \mathcal{O}_C$ must have Clifford index no bigger than the expected maximal g^r_d by Conjecture 2, the upper bound on $\gamma(r', d')$ comes from Remark 1.2. The other bounds are obtained in the same way as for L3.

Remark 1.12. As stated above, computations show that L1 holds for every expected maximal locus up to large genus.

We note that L2 and L3 do not always hold. The first genus where L3 fails is $g = 56$, where L3 fails for the lattices $\Lambda^r_{g,d} = \Lambda^2_{56,39}$ and $\Lambda^{r'}_{g,d'} = \Lambda^6_{56,49}$; indeed, in attempting to check whether $\mathcal{M}^2_{56,39}$ can be contained in $\mathcal{M}^3_{56,44}$, a g^3_{44} on a curve $C \in |H|$ for a very general $(S,H) \in \mathcal{K}^2_{56,39}$ has a potential Donagi–Morrison lift M of type g^6_{49}. However, $\Lambda^2_{56,39} \cong \Lambda^6_{56,49}$, and so L3 does not hold. In this case, because $\rho(56,2,39) = -1$ and $\rho(56,3,44) = -4$, we clearly have $\mathcal{M}^2_{56,39} \not\subseteq \mathcal{M}^3_{56,44}$. Hence the failure of L3 does not necessarily obstruct our program to prove that Conjecture 2 implies Conjecture 1.

The next genus where L3 fails is $g = 89$, where the locus $\mathcal{M}^3_{89,69}$ could possibly be contained in $\mathcal{M}^4_{89,75}$ or $\mathcal{M}^5_{89,79}$. This is because line bundles of type g^3_{36} and g^4_{75} have a potential Donagi–Morrison lift M of type g^{10}_{85}, and the lattice $\langle H, M \rangle = \Lambda^{10}_{89,85}$ is isomorphic to $\Lambda^3_{89,69}$, so that L3 does not hold. In this example, $\mathcal{M}^3_{89,69}$ has codimension 3 in $\mathcal{M}_{89}$, whereas $\mathcal{M}^4_{89,75}$ and $\mathcal{M}^5_{89,79}$ both have codimension 1, hence the codimensions of the loci do not rule out the possibility that $\mathcal{M}^3_{89,69}$ is not maximal. Thus in genus 89, Conjecture 2 together with L2 is not sufficient to imply Conjecture 1 without additional techniques.

We note that below genus 200, except for genus $89, 91, 92, 145, 153$, and 190, L2 holds, and thus Conjecture 2 implies Conjecture 1.

Proposition 1.13. *Let* $\mathcal{M}^r_{g,d}$ *and* $\mathcal{M}^{r'}_{g,d'}$ *be two expected maximal Brill–Noether loci. Suppose* (S,H) *is a polarized K3 surface with* $\mathrm{Pic}(S) = \Lambda^r_{g,d}$*, and* $L \otimes \mathcal{O}_C$ *is a* g^r_d*. If the Donagi–Morrison* *conjecture (Conjecture 2) holds for $g^{r'}_{d'}$ on C and L3 holds for the pair $(\Lambda^r_{g,d}, \Lambda^{r'}_{g,d'})$, then $\mathcal{M}^r_{g,d} \nsubseteq \mathcal{M}^{r'}_{g,d'}$. In particular, if Conjecture 2 and L2 hold for all expected maximal g^r_d in genus g, then Conjecture 1 holds in genus g.*

Proof. The condition L3 implies that $\mathrm{Pic}(S)$ cannot admit any potential Donagi–Morrison lift of the $g^{r'}_{d'}$. Hence the existence of a $g^{r'}_{d'}$ on C contradicts the Donagi–Morrison conjecture. Therefore C has no $g^{r'}_{d'}$, as was to be shown. □

To state a related question, we need a simple definition.

Definition 1.14. We define the *special Clifford index* of C as

$$\widetilde{\gamma}(C) := \{\gamma(A) \mid \rho(A) < 0,\, h^0(C, A) \geq 2, \text{ and } h^1(C, A) \geq 2\}.$$

We say a Brill–Noether special line bundle A on C *computes* the special Clifford index if $\gamma(A) = \widetilde{\gamma}(C)$.

Lelli-Chiesa's lifting result [27, Theorem 4.2] provides a lift of Brill–Noether special line bundles computing the Clifford index. A similar result for line bundles computing the special Clifford index of the curve together with L1 would imply that $\mathcal{M}^r_{g,d} \nsubseteq \mathcal{M}^{r'}_{g,d'}$ for $\gamma(r, d) \geq \gamma(r', d')$. We are left with three questions to which positive answers would imply parts of Conjecture 1.

Question 1. When L1 or L2 fail, can the Brill–Noether loci be distinguished in another way?

Question 2. Under what conditions does a line bundle computing the special Clifford index of a curve C lift to a line bundle on S?

Question 3. Does the Donagi–Morrison conjecture hold for expected maximal g^r_ds?

We note that the work on Brill–Noether theory for fixed gonality, if it were extended to higher rank, could provide another approach to distinguishing Brill–Noether loci that is complementary to the Donagi–Morrison lifting approach.

2. Stability of Sheaves on K3 Surfaces

We recall the notions of stability and slope stability of torsion free coherent sheaves on a polarized K3 surface (S, H) and Harder–Narasimhan (HN) and Jordan–Hölder (JH) filtrations. Let E be a torsion free coherent sheaf on (S, H) The *slope* of E is $\mu_H(E) := \frac{c_1(E).H}{\mathrm{rk}(E)}$. A torsion free coherent sheaf is called slope stable or μ-stable (μ-semistable) if $\mu_H(F) < \mu_H(E)$ (respectively, $\mu_H(F) \leq \mu_H(E)$) for all coherent sheaves $F \subseteq E$ with $0 < \mathrm{rk}(F) < \mathrm{rk}(E)$. We define the *normalized Hilbert polynomial* of E to be

$$p(E, n) := \frac{\chi(E \otimes H^n)}{\mathrm{rk}(E)} = \frac{H^2}{2!}n^2 + \mu(E)n + \frac{\chi(E)}{\mathrm{rk}(E)}$$

where the second equality follows from Riemann–Roch. We say E is (Gieseker) stable (semistable) if $p(F, n) < p(E, n)$ (respectively, $p(F, n) \leq p(E, n)$) for all proper subsheaves $F \subsetneq E$, where for two polynomials $f(n), g(n)$ we say $f(n) < g(n)$ ($f(n) \leq g(n)$) if this is true for $n \gg 0$.

We have the following implications for a torsion free coherent sheaf E

$$\mu\text{-stable} \implies \text{stable} \implies \text{semistable} \implies \mu\text{-semistable}.$$

Every torsion free coherent sheaf E has a unique Harder–Narasimhan filtration, which is an increasing filtration

$$0 = HN_0(E) \subset HN_1(E) \subset \cdots \subset HN_\ell(E) = E,$$

such that the factors $gr_i^{HN}(E) = HN_i(E)/HN_{i-1}(E)$ for $i = 1, \ldots, \ell$ are torsion free semistable sheaves with normalized Hilbert polynomials $p_i = p(gr_i^{HN}(E), n)$ satisfying

$$p_{\max} = p_1 > \cdots > p_\ell = p_{\min}.$$

In particular, we see that $\mu(gr_1^{HN}(E)) > \mu(gr_2^{HN}(E)) > \cdots > \mu(gr_\ell^{HN}(E))$. If E is a vector bundle, the sheaves $HN_i(E)$ are locally free. We also have $\mu(HN_1(E)) > \mu(HN_2(E)) > \cdots > \mu(E)$.

Likewise, every (μ)-semistable sheaf E has a Jordan–Hölder filtration, which is an increasing filtration

$$0 = JH_0(E) \subset JH_1(E) \subset \cdots \subset JH_\ell(E) = E,$$

such that the factors $gr_i^{JH}(E) = JH_i(E)/JH_{i-1}(E)$ for $i = 1, \ldots, \ell$ are torsion free stable sheaves with normalized Hilbert polynomial $p(E, n)$. In particular, $\mu(E) = \mu(gr_i^{JH}(E))$ for all i. The JH filtration is not uniquely determined, however the associated graded object $gr^{JH}(E) = \bigoplus_i gr_i^{JH}(E)$ is uniquely determined by E.

We also briefly recall some facts about the moduli space of stable and semistable sheaves on K3 surfaces from [18]. For a sheaf E on (S, H), the *Mukai vector* is given by

$$v(E) := ch(E)\sqrt{td(S)} = (\mathrm{rk}(E),\ c_1(E),\ ch_2(E) + \mathrm{rk}(E)) = (\mathrm{rk}(E),\ c_1(E),\ \chi(E) - \mathrm{rk}(E)),$$

considered as an element in $H^*(S, \mathbb{Z})$. For a fixed Mukai vector v, the moduli space of semistable sheaves with Mukai vector v is denoted $M(v)$, and the open (possibly empty) subscheme of stable sheaves is denoted $M(v)^s \subset M(v)$. The Mukai pairing is given by

$$\langle v(E), v(F)\rangle := \chi(E, F) = \sum_i (-1)^i \operatorname{Ext}^i(E, F) = -\int_S v(E)^* \wedge v(F),$$

where for $v(E) = v^0 + v^2 + v^4 \in H^i(S, \mathbb{Z})$ with $v^i \in H^i(S, \mathbb{Z})$, $v(E)^* := v^0 - v^2 + v^4$. We recall that the space of stable sheaves with Mukai vector v, $M(v)^s$ is either empty or a smooth quasi-projective variety of dimension $2 + \langle v, v\rangle$.

3. Lazarsfeld–Mukai Bundles and Lifting

We briefly recall some facts about Lazarsfeld–Mukai bundles (LM bundles) and state a few useful facts that motivate our proof. Let $\iota\colon C \hookrightarrow S$ be a smooth irreducible curve of genus g in S and A a basepoint free line bundle on C of type g_d^r. We define a bundle $F_{C,A}$ on S via the short exact sequence

$$0 \longrightarrow F_{C,A} \longrightarrow H^0(C, A) \otimes \mathcal{O}_S \xrightarrow{ev} \iota_*(A) \longrightarrow 0.$$

Dualizing gives $E_{C,A} := F_{C,A}^\vee$ (the LM bundle associated to A on C) sitting in the short exact sequence

$$0 \longrightarrow H^0(C, A)^\vee \otimes \mathcal{O}_S \longrightarrow E_{C,A} \longrightarrow \iota_*(\omega_C \otimes A^\vee) \longrightarrow 0;$$

whereby the following facts about the LM bundle $E_{C,A}$ are readily proved.

Proposition 3.1. *Let $E_{C,A}$ be a LM bundle associated to a basepoint free line bundle A of type g_d^r on $C \subset S$, then:*

- $\det E_{C,A} = c_1(E_{C,A}) = [C]$ *and* $c_2(E_{C,A}) = \deg(A)$*;*
- $\mathrm{rk}(E_{C,A}) = r + 1$ *and $E_{C,A}$ is globally generated off the base locus of* $\iota_*(\omega_C \otimes A^\vee)$*;*
- $h^0(S, E_{C,A}) = h^0(C, A) + h^0(C, \omega_C \otimes A^\vee) = 2r + 1 + g - d = g - (d - 2r) + 1$*;*
- $h^1(S, E_{C,A}) = h^2(S, E_{C,A}) = 0$*,* $h^0(S, E_{C,A}^\vee) = h^1(S, E_{C,A}^\vee) = 0$*;*
- $\chi(F_{C,A} \otimes E_{C,A}) = 2(1 - \rho(g, r, d))$*.*

A vector bundle E is called *simple* if $\mathrm{End}(E)$ is a division algebra. Over an algebraically closed field, this is equivalent to $h^0(E^\vee \otimes E) = 1$. Thus we see that $E_{C,A}$ is non-simple if $\rho(g, r, d) < 0$.

In [27], generalized LM bundles are defined and prove useful in lifting special line bundles on a curve $C \in |H|$ to a line bundle on the polarized K3 surface (S, H).

Definition 3.2. Let C be a curve and $A \in \mathrm{Pic}(C)$. The linear system $|A|$ is called *primitive* if both A and $\omega_C \otimes A^\vee$ are basepoint free.

Definition 3.3 ([27] Definition 1)**.** A torsion free coherent sheaf E on S with $h^2(S, E) = 0$ is called a *generalized Lazarsfeld–Mukai bundle* (gLM bundle) of type (I) or (II), respectively, if

(I) E is locally free and generated by global sections off a finite set;
or
(II) E is globally generated.

Remark 3.4 ([27] Remark 1)**.** If conditions (I) and (II) of Definition 3.3 are both satisfied, then E is the LM bundle associated with a smooth irreducible curve $C \subset S$ and a primitive linear series (A, V) on C, i.e. $E = E_{C,(A,V)}$, where $E_{C,(A,V)}$ is the dual of the kernel of the evaluation map $V \otimes \mathcal{O}_S \to A$. Furthermore, $V = H^0(C, A)$ if and only if $h^1(S, E) = 0$, in which case E is just the LM bundle associated to A.

Definition 3.5. Let E be a gLM bundle. The *Clifford Index of E* is:

$$\gamma(E) := c_2(E) - 2(\mathrm{rk}(E) - 1).$$

Remark 3.6. For the LM bundle $E_{C,A}$ for a smooth curve $C \subset S$ and A a g^r_d on C, one has $\gamma(E_{C,A}) = \gamma(A)$ by Proposition 3.1.

Lemma 3.7 ([27] Corollary 2.5)**.** *Let E be a gLM bundle of rank r and $c_1(E)^2 > 0$. Then, $\gamma(E) \geq 0$. Furthermore, $\gamma(E) = 0$ only in the following cases:*

(a) *$r = 1$ and E is a globally generated line bundle;*
(b) *$E = E_{C,\omega_C}$ for some smooth irreducible curve $C \subset S$ of genus $g = r \geq 2$;*
(c) *$r > 1$ and $E = E_{C,(r-1)g^1_2}$ for some smooth hyperelliptic curve $C \subset S$ of genus $g > r$.*

Lemma 3.8. *Let $N \in \mathrm{Pic}(S)$ be nontrivial and globally generated with $h^0(S, N) \neq 0$. Let $E = E_{C,A}$ and suppose we have a short exact sequence*

$$0 \longrightarrow N \longrightarrow E \longrightarrow E/N \longrightarrow 0$$

with E/N torsion free. Then E/N satisfies $h^1(S, E/N) = h^2(S, E/N) = 0$. If A is primitive, then E/N is a gLM bundle of type (II). If we further assume that E/N is locally free, then it is a LM bundle for a smooth irreducible curve $D \in |H - N|$. If A is not primitive and E/N is assumed locally free, then E/N is a gLM bundle of type (I). In any of the above cases, we have

- $c_1(E/N) = H - N$;
- $c_2(E/N) = d + N^2 - H.N$;
- $\gamma(E/N) = \gamma(E_{C,A}) + N^2 - H.N + 2$.

Proof. If A is primitive, we see that E/N is globally generated as E is globally generated. From the long exact sequence in cohomology, and noting that $h^2(S, N) = h^1(S, E) = h^2(S, E) = 0$, we see that $h^1(S, E/N) = h^2(S, E/N) = 0$. Thus E/N is a gLM bundle of type (II). If E/N is assumed to be locally free, then as in Remark 3.4, $E/N = E_{D,B}$ is the LM bundle associated to a smooth irreducible curve $D \subset S$ and a line bundle B on D. Finally, if A is not primitive, then E/N is globally generated off a finite set as it is the quotient of E, which is also globally generated off a finite subset. Thus E/N is a gLM of type (I).

Applying Whitney's formula to the exact sequence, we see that

$$1 + c_1(E) + c_2(E) = (1 + c_1(E/N) + c_2(E/N))\,(1 + N)\,,$$

hence $c_1(E/N) = H - N$ and $c_2(E/N) = d + N^2 - H.N$. Finally, as $\gamma(E/N) = c_2(E/N) - 2(\mathrm{rk}(E/N) - 1)$ and $\mathrm{rk}(E/N) = \mathrm{rk}(E) - 1 = (r + 1) - 1 = r$, it follows that

$$\gamma(E/N) = d + N^2 - H.N - 2(r - 1) = d - 2r + N^2 - H.N + 2 = \gamma(E) + N^2 - H.N + 2.$$

□

Remark 3.9. If A is of type g^r_d and $L = H - N$ is a lift of A with $L^2 = 2r - 2$, then the last equality gives $\gamma(E/N) = \gamma(A) + (2r - 2) - d + 2 = 0$.

We recall a few lemmas which show when a linear series on a curve $C \in |H|$ is the restriction of a line bundle L on S.

Lemma 3.10. *Let (S, H) be a polarized K3 surface of genus $g \geq 2$, $C \in |H|$ be a smooth irreducible curve, and L a globally generated line bundle on S such that $L|_C$ is a g^r_d with $c_1(L).C = d < 2g-2$. Then if $h^1(S, L) = 0$, we have $L^2 = 2r - 2 - 2h^1(S, L(-C))$.*

Proof. Since H is basepoint free and $c_1(L(-C)).C = d - (2g - 2) < 0$, we have $h^0(S, L(-C)) = 0$, as in the proof of [22, Proposition 2.1]. We now consider the short exact sequence for a divisor $C \subset S$ tensored with L,

$$0 \longrightarrow L(-C) \longrightarrow L \longrightarrow L|_C \longrightarrow 0.$$

By Riemann-Roch on C we have $h^1(S, L|_C) = h^1(C, L|_C) = r - d + g$, and as $h^1(S, L) = h^2(S, L) = 0$, the long exact sequence in cohomology and Serre duality give $h^2(S, L(-C)) = h^0(S, L(-C)^\vee) = r - d + g$. By Riemann-Roch on S, we have

$$h^0(S, L(-C)^\vee) - h^1(S, L(-C)) = 2 + \frac{c_1(L(-C))^2}{2} = 2 + \frac{c_1(L)^2 - 2d + 2g - 2}{2} = 1 - d + g + \frac{c_1(L)^2}{2}$$

thus $c_1(L)^2 = 2r - 2 - 2h^1(S, L(-C))$. □

Corollary 3.11. *Let (S, H) be a polarized K3 surface of genus $g \geq 2$, A a complete g^r_d on a smooth $C \in |H|$. Let $N \in \mathrm{Pic}(S)$ be a line bundle with $h^0(S, N) \geq 2$ and $h^1(S, N) = 0$. Assume $H \otimes N^\vee$ is globally generated, satisfies $h^1(S, H \otimes N^\vee) = 0$, and is a lift of A. Then $c_1(H \otimes N^\vee)^2 = 2r - 2$.*

Proof. We have $h^1(S, N) = 0$. Hence as $N^\vee = H \otimes N^\vee \otimes H^\vee$, Serre duality gives $0 = h^1(S, N^\vee) = h^1(S, H \otimes N^\vee(-C))$. Thus Lemma 3.10 shows that $(H - N)^2 = 2r - 2$. □

Remark 3.12 ([27] Remark 6)**.** The proof [27, Lemma 4.1] shows that as soon as we have a nontrivial $N \in \mathrm{Pic}(S)$ with $h^0(S, N) \neq 0$ and an injection $N \hookrightarrow E_{C,A}$, we have $h^0(S, \iota_*(A) \otimes (H \otimes N^\vee) \otimes \mathcal{O}_C) = h^0(C, A^\vee \otimes (H \otimes N^\vee)|_C) \neq 0$, i.e., the linear series $|A|$ is contained in $|(H \otimes N^\vee)|_C|$. We also note that if $h^1(S, N) = 0$, then

$$H^0(C, (H \otimes N^\vee) \otimes \mathcal{O}_C) = H^0(S, H \otimes N^\vee)|_C.$$

Lemma 3.13. *Let N be a line bundle and $0 \to N \to E \to E/N \to 0$ be a short exact sequence of coherent sheaves on a polarized K3 surface (S, H), where E/N is stable, $\mathrm{rk}(E) = r+1$, $c_1(E) = H$, $c_1(E)^2 = 2g - 2 \geq 0$. If $h^0(S, N) < 2$, then $c_2(E) \geq \frac{g(r-1)}{r} + \frac{2g-2}{r(r+1)} + r + \frac{1}{r}$.*

Proof. Since $\mu(N) \geq \mu(E) \geq 0$, we have $h^2(S, N) = 0$. Therefore if $h^0(S, N) < 2$ we have $c_1(N)^2 \leq -2$. Hence

$$c_1(E/N)^2 + 2c_1(N).c_1(E/N) = c_1(E)^2 - c_1(N)^2 \geq 2g - 2 + 2 = 2g$$

and
$$c_1(E/N).c_1(N) = c_1(N).(c_1(E) - c_1(N)) \geq \frac{2g-2}{r+1} + 2,$$
where the last inequality comes from the fact that $\mu(N) \geq \mu(E)$. Thus $\frac{c_1(E/N)^2}{2} \geq g - c_1(N).c_1(E/N)$.

Furthermore, since E/N is stable of rank r, the dimension of the moduli space of stable sheaves with Mukai vector $\nu(E/N)$, $M^s_{\nu(E/N)}$, has non-negative dimension. Thus $2rc_2(E/N) - (r-1)c_1(E/N)^2 - 2(r^2-1) \geq 0$, and we have $c_2(E/N) \geq r + \frac{1}{r} + \left(\frac{r-1}{2r}\right) c_1(E/N)^2$.

We now calculate $c_2(E) = c_1(E/N).c_1(N) + c_2(E/N) \geq \frac{g(r-1)}{r} + \frac{2g-2}{r(r+1)} + r + \frac{1}{r}$, as desired. □

We present a version of [26, Proposition 7.4] which motivates our proof strategy below.

Proposition 3.14. *Let (S, H) be a polarized K3 surface and A be a complete basepoint free g^r_d on a smooth irreducible curve $C \in |H|$ with $r \geq 2$ and let $E = E_{C,A}$. Suppose that E sits in a short exact sequence*
$$0 \longrightarrow N \longrightarrow E \longrightarrow E/N \longrightarrow 0$$
for some line bundle N and $c_2(E) = d < \frac{g(r-1)}{r} + \frac{2g-2}{r(r+1)} + r + \frac{1}{r}$. If E/N is stable, or E/N is semistable and there are no elliptic curves on S, then $|A|$ is contained in the restriction to C of the linear system $|H \otimes N^\vee|$ on S. Moreover, $H \otimes N^\vee$ is adapted to $|H|$ and $\gamma(H \otimes N^\vee \otimes \mathcal{O}_C) \leq d - r - 3$.

Proof. By the previous lemma, $h^0(S, N) \geq 2$. We also have $h^0(S, \det E/N) \geq 2$ from [26, Lemma 3.3]. We note that $(E/N)^{\vee\vee}$ is globally generated off a finite set and
$$h^i(S, (E/N)^{\vee\vee}) = h^i(S, E/N) = 0 \text{ for } i = 1, 2.$$
Since $\det E/N = \det(E/N)^{\vee\vee}$ is basepoint free and nontrivial, $\det E/N$ is nef, thus $c_1(E/N)^2 \geq 0$. If $h^1(S, \det E/N) \neq 0$, then $c_1(E/N)^2 = 0$ by Saint-Donat. By [14, Proposition 1.1], there is a smooth elliptic curve $\Sigma \subset S$ such that $(E/N)^{\vee\vee} = \mathcal{O}(\Sigma)^{\oplus 3}$. This contradicts the stability of E/N (or the non-existence of elliptic curves on S), thus we must have $c_1(E/N)^2 \geq 2$ (hence $c_2(E/N) \geq r + 1$) and $h^1(S, \det E/N) = 0$. This ensures that $h^0(C, \det E/N \otimes \mathcal{O}_C) = h^0(C, H \otimes N^\vee \otimes \mathcal{O}_C)$ does not depend on the curve $C \in |H|_s$. Hence $\det E/N = H \otimes N^\vee$ is adapted to $|H|$. We calculate
$$\begin{aligned}
\gamma(\det E/N \otimes \mathcal{O}_C) &= c_1(E/N).c_1(E) - 2h^0(C, \det E/N \otimes \mathcal{O}_C) + 2 \\
&= c_1(E/N)^2 + c_1(N).c_1(E/N) - 2h^0(C, \det E/N \otimes \mathcal{O}_C) + 2 \\
&\leq c_1(E/N)^2 - 2h^0(S, \det E/N) + c_1(N).c_1(E/N) + 2 \\
&= -2h^1(S, \det E/N) - 4 + c_1(N).c_1(E/N) + 2 \\
&= d - c_2(E/N) - 2 \leq d - r - 3.
\end{aligned}$$
The claim that $|A|$ is contained in $|H \otimes N^\vee \otimes \mathcal{O}_C|$ is proved in the same way as in [27, Lemma 4.1]. □

Remark 3.15. In the above proposition, if A is of type g^3_d, then $\gamma(H \otimes N^\vee \otimes \mathcal{O}_C) \leq d - r - 3 = \gamma(A)$. However, as soon as $r \geq 4$, then $\gamma(H \otimes N^\vee \otimes \mathcal{O}_C)$ may be bigger than $\gamma(A)$. However, Lelli-Chiesa proves in [27, Propositioon 5.1] that $\gamma(H \otimes N^\vee \otimes \mathcal{O}_C) \leq \gamma(A)$ whenever $N \subset E$ is a saturated subsheaf and $h^1(S, N) = 0$.

4. Filtrations of Lazarsfeld–Mukai Bundles of Rank 4

Throughout this section, (S, H) is a polarized K3 surface of genus g, $C \in |H|$ is a smooth irreducible curve, A is a line bundle of type g^3_d on C, and $E = E_{C,A}$ is the LM bundle corresponding to A. Given E, we can take its JH filtration or take its HN filtration, further take JH filtrations of the properly semistable factors, lift the JH factors and expand the HN filtration of E to arrive at a *terminal filtration* such that all quotients are stable sheaves. We enumerate all the possibilities listing a filtration by the ranks of the terms, i.e., a filtration of type $0 \subset 1 \subset 4$ is a filtration $0 \subset N \subset E$ where $\mathrm{rk}(N) = 1$.

The terminal filtrations correspond to flags of E where each quotient is stable, hence the terminal filtrations are

$$\begin{aligned}
&0 \subset 1 \subset 4, \quad 0 \subset 2 \subset 4, \quad 0 \subset 3 \subset 4,\\
&0 \subset 1 \subset 2 \subset 4, \quad 0 \subset 1 \subset 3 \subset 4, \quad 0 \subset 2 \subset 3 \subset 4,\\
&0 \subset 1 \subset 2 \subset 3 \subset 4.
\end{aligned}$$

In order to apply Proposition 3.14, we want to show that given the g^3_d, E must have a terminal filtration of type $0 \subset 1 \subset 4$. In all other cases, we want to find a lower bound on $d = c_2(E)$. To this end, we find a bound for $c_2(E)$ in terms of the intersections of the Chern roots of the LM bundle E. We begin by noting a few general bounds, and then deal with each filtration.

We slightly generalize the proof of [26, Lemma 4.1].

Proposition 4.1. *Let E a LM bundle with $c_1(E) = H$ and $\mu(E) = \frac{g-1}{2} > 0$ sitting in an exact sequence*

$$0 \longrightarrow M \longrightarrow E \longrightarrow M_1 \longrightarrow 0$$

where M and M_1 are coherent sheaves. Suppose that the general smooth curve $C \in |H|$ has (constant) Clifford index $\gamma = \gamma(C)$. Then one has $c_1(M).c_1(M_1) \geq \gamma + 2$.

Proof. We write $\mu(F) = \mu_H(F)$. Since M_1 is a quotient of E, it is globally generated off a finite set of points. Moreover, we have $h^2(S, M_1) = 0$, thus $h^0(S, \det M_1) \geq 2$ by [26, Lemma 3.3] as the vector bundle $M_1^{\vee\vee}$ is globally generated off a finite number of points and $\det(M_1) := \det(M_1^{\vee\vee})$. As in [26, Lemma 3.2], we see that $\det M_1$ is basepoint free and nontrivial, thus $\mu(\det M_1) > 0$, $\mu(M) > 0$. Hence as $\mu(\det M) \geq \mu(M) > 0$, [26, Proposition 3.1] shows that $h^2(S, \det M_1) = 0$, $h^2(S, \det M) = 0$, and that $\det M_1$ is nef whereby $c_1(M_1)^2 \geq 0$.

Furthermore, as

$$\mu(M) = \frac{c_1(M).c_1(E)}{\mathrm{rk}(M)} = \frac{c_1(M).(c_1(M) + c_1(M_1))}{\mathrm{rk}(M)} \geq \frac{g-1}{2},$$

we have $c_1(M).c_1(M_1) \geq \mathrm{rk}(M)\frac{g-1}{2} - c_1(M)^2$. Since $h^2(S, \det M) = 0$, we note that

$$h^0(S, \det M) \geq h^0(S, \det M) - h^1(S, \det M) = \chi(\det M) = 2 + \frac{c_1(M)^2}{2}.$$

Therefore, if $2 > h^0(S, \det M)$, then $c_1(M)^2 \leq -2$, and thus

$$c_1(M).c_1(M_1) \geq \mathrm{rk}(M)\frac{g-1}{2} + 2 \geq \mathrm{rk}(M)\gamma + 2 \geq \gamma + 2$$

as $\mathrm{rk}(M) \geq 1$.

Hence from now on we assume that $h^0(S, \det M) \geq 2$. Since $\omega_C \otimes (\det M_1)^\vee \otimes \mathcal{O}_C = \det M \otimes \mathcal{O}_C$, the line bundle $\det M_1 \otimes \mathcal{O}_C$ contributes to $\gamma(C)$. Tensoring the short exact sequence for $\mathcal{O}_C$ with $\det M_1$ gives

$$0 \longrightarrow \det M^\vee \longrightarrow \det M_1 \longrightarrow \det M_1 \otimes \mathcal{O}_C \longrightarrow 0\ ,$$

which gives $h^0(C, \det M_1 \otimes \mathcal{O}_C) \geq h^0(S, \det M_1)$. It follows that

$$\begin{aligned}
\gamma(\det M_1 \otimes \mathcal{O}_C) &= c_1(M_1).(c_1(M) + c_1(M_1)) - 2h^0(C, \det M_1 \otimes \mathcal{O}_C) + 2\\
&\leq c_1(M_1)^2 + c_1(M).c_1(M_1) - 2\chi(\det M_1) - 2h^1(S, \det M_1) + 2\\
&= -2 + c_1(M).c_1(M_1) - 2h^1(S, \det M_1).
\end{aligned}$$

By assumption, we have $\gamma(\det M_1 \otimes \mathcal{O}_C) \geq \gamma$, thus $c_1(M).c_1(M_1) \geq \gamma + 2 + 2h^1(S, \det M_1) \geq \gamma + 2$, as desired. □

Remark 4.2. It follows from the second half of the proof that if M and M_1 are coherent sheaves such that $c_1(M) + c_1(M_1) = c_1(E)$, $\det M_1 \otimes \mathcal{O}_C$ (hence also $\det M \otimes \mathcal{O}_C$) contributes to $\gamma(C)$, and $h^2(S, \det M_1) = 0$ (or $h^2(S, \det M) = 0$), then $c_1(M).c_1(M_1) \geq \gamma(C) + 2 + 2h^1(S, \det M_1) \geq \gamma(C) + 2$ (or $c_1(M).c_1(M_1) \geq \gamma(C) + 2 + 2h^1(S, \det M) \geq \gamma(C) + 2$).

Proposition 4.3. *Let (S, H) be a polarize K3 surface, $C \in |H|$ a smooth irreducible curve, A a basepoint free line bundle on A of type g^3_d, and $E = E_{C,A}$. Suppose E sits in an exact sequence*

$$0 \longrightarrow M \longrightarrow E \longrightarrow E/M \longrightarrow 0 ,$$

where M and E/M are coherent torsion free sheaves on S and $\mu(M) \geq \mu(E) \geq \mu(E/M)$. If $\mathrm{rk}(M) \geq \mathrm{rk}(E/M)$, then $c_1(M)^2 \geq c_1(E/M)^2$. And if $\mathrm{rk}(M) > \mathrm{rk}(E/M)$, then $c_1(M)^2 > c_1(E/M)^2$. In particular, $\det(E/M) \otimes \mathcal{O}_C$ contributes to $\gamma(C)$.

Proof. As in Proposition 4.1, we see $h^0(S, \det E/M) \geq 2$, $\mu(E/M) > 0$, $\det(E/M)$ is nef, and $h^2(S, \det M) = 0$. Since $h^0(S, \det E/M) \geq 2$, it remains to show that $h^0(S, \det M) \geq 2$.

We observe that

$$c_1(M)^2 + c_1(M).c_1(E/M) = \mathrm{rk}(M)\mu(M) \geq \mathrm{rk}(E/M)\mu(E/M) = c_1(E/M)^2 + c_1(M).c_1(E.M)$$

whence $c_1(M)^2 \geq c_1(E/M)^2 \geq 0$ as $\det(E/M)$ is nef.

Since $h^2(S, \det M) = 0$, we have $h^0(S, \det M) \geq \chi(\det M) = 2 + \frac{c_1(M)^2}{2}$. Thus as $c_1(M)^2 \geq 0$, $\det(E/M) \otimes \mathcal{O}_C$ contributes to $\gamma(C)$. □

For each terminal filtration not of the form $0 \subset 1 \subset 4$, we find a lower bound for $d = c_2(E)$. That is whenever E does not have a maximal destabilizing sub-line bundle, we find that d must be large.

4.1. Filtration $2 \subset 4$.

We assume E is unstable with terminal filtration $0 \subset M \subset E$ with M and $M_1 = E/M$ stable rank 2 torsion free sheaves. Thus E sits in an exact sequence of the form

$$0 \longrightarrow M \longrightarrow E \longrightarrow M_1 \longrightarrow 0 .$$

We have

$$\mu(M) \geq \mu(E) = \frac{g-1}{2} \geq \mu(M_1) \tag{1}$$

$$d = c_2(E) = c_1(M).c_1(M_1) + c_2(M) + c_2(M_1) \tag{2}$$

Lemma 4.4. *Suppose $C \in |H|_s$ has Clifford index $\gamma = \gamma(C)$. Then if E is as above, we have $d \geq \frac{\gamma}{2} + 4 + \frac{g-1}{2}$.*

Proof. From Proposition 4.1 and Proposition 4.3, we see that $c_1(M).c_1(M_1) \geq \gamma + 2$. Stability of M and M_1 give $-2 \leq \langle \nu(M_{(1)}), \nu(M_{(1)}) \rangle = 4c_2(M_{(1)}) - c_1(M_{(1)})^2 - 8$, thus $c_2(M_{(1)}) \geq \frac{3}{2} + \frac{c_1(M_{(1)})^2}{4}$.

We have

$$\frac{c_1(M)^2 + c_1(M_1)^2}{4} + \frac{c_1(M).c_1(M_1)}{2} = \frac{\mu(M) + \mu(M_1)}{2} = \frac{(c_1(M) + c_1(M_1))^2}{4} = \mu(E) = \frac{g-1}{2}.$$

We now calculate

$$\begin{aligned}
d &= c_1(M).c_1(M_1) + c_2(M) + c_2(M_1) + l(\xi) \\
&\geq c_1(M).c_1(M_1) + 3 + \frac{c_1(M)^2 + c_1(M_1)^2}{4} \\
&= c_1(M).c_1(M_1) + 3 + \frac{g-1}{2} - \frac{c_1(M)/c_1(M_1)}{2} \\
&\geq \frac{\gamma+2}{2} + 3 + \frac{g-1}{2}.
\end{aligned}$$

as claimed. □

4.2. **Filtration $3 \subset 4$.** We assume $E = E_{C,A}$ is unstable with terminal filtration $0 \subset M \subset E$ with M a stable rank 3 torsion free sheaf. Thus E sits in an extension

$$0 \longrightarrow M \longrightarrow E \longrightarrow N \otimes I_\xi \longrightarrow 0$$

where N is a line bundle and I_ξ is the ideal sheaf of a 0-dimensional subscheme $\xi \subset S$ of length $l(\xi) = d - c_1(M).c_1(N)$. We have

$$\mu(M) \geq \mu(E) = \frac{g-1}{2} \geq \mu(N) \tag{3}$$

$$c_1(H) = c_1(E) = c_1(M) + c_1(N) \tag{4}$$

$$d = c_2(E) = c_1(N).c_1(M) + c_2(M) + l(\xi) \tag{5}$$

Lemma 4.5. *Suppose $C \in |H|_s$ has Clifford index $\gamma = \gamma(C)$. Then if E is as above, we have $d \geq \frac{2}{3}(\gamma+2) + \frac{g}{2} + \frac{13}{6}$.*

Proof. From Proposition 4.1 and Proposition 4.3, we see that $c_1(N).c_1(M) \geq \gamma + 2$.

As M is stable, we have $-2 \leq \langle \nu(M), \nu(M) \rangle = 6c_2(M) - 2c_1(M)^2 - 18$, thus $c_2(M) \geq \frac{8+c_1(M)^2}{3}$. Thus

$$\begin{aligned}
d &= c_1(N).c_1(M) + c_2(M) + l(\xi) \\
&\geq c_1(N).c_1(M) + \frac{c_1(M)^2}{3} + \frac{8}{3} \\
&\geq c_1(N).c_1(M) + \frac{g-1}{2} - \frac{c_1(N).c_1(M)}{3} + \frac{8}{3} \\
&\geq \frac{2}{3}(\gamma+2) + \frac{g}{2} + \frac{13}{6},
\end{aligned}$$

as desired. □

4.3. **Filtration $1 \subset 2 \subset 4$.** We assume E has a terminal filtration $0 \subset N \subset M \subset E$ with $\mathrm{rk}(N) = 1$, $\mathrm{rk}(M) = 2$, and $E/M = M_1$ a stable torsion free sheaf. Furthermore, we have

$$\mu(N) \geq \mu(M) \geq \mu(E) = \frac{g-1}{2} \geq \mu(M_1) \tag{6}$$

$$\mu(M) \geq \mu(M/N) \geq \mu(E/N) \geq \mu(M_1) \tag{7}$$

(8)
$$d = c_2(E) = c_2(M) + c_2(M_1) + c_1(M).c_1(M_1) = c_1(N).c_1(M/N) + c_1(N).c_1(M_1) + c_1(M/N).c_1(M_1) + c_2(M_1)$$

Moreover, as M_1 is stable, we have

$$-2 \leq \langle \nu(M_1), \nu(M_1) \rangle = c_1(M_1)^2 - 4\chi(M_1) + 8 = 4c_2(M_1) - c_1(M_1)^2 - 8$$

thus $c_2(M_1) \geq \frac{3}{2} + \frac{c_1(M_1)^2}{4}$. We find below that $c_1(M_1)^2 \geq 0$ therefore we have

$$d \geq 2 + c_1(N).c_1(M/N) + c_1(N).c_1(M_1) + c_1(M/N).c_1(M_1). \tag{9}$$

Lemma 4.6. *Suppose E is as above. Then $\det M_1 \otimes \mathcal{O}_C$ contributes to $\gamma(C)$ and one of the following occurs:*

(a) *$N \otimes \mathcal{O}_C$ and $(M/N) \otimes \mathcal{O}_C$ contribute to $\gamma(C)$;*
(b) *$c_1(N).(c_1(M_1) + c_1(M/N)) \geq \frac{g-1}{2} + 2$ and either $(M/N) \otimes \mathcal{O}_C$ contributes to $\gamma(C)$ or $c_1(M/N).(c_1(N) + c_1(M_1)) \geq g$;*
(c) *$N \otimes \mathcal{O}_C$ contributes to $\gamma(C)$ and $c_1(M/N).(c_1(N) + c_1(M_1)) \geq 2 + \frac{c_1(M).c_1(M_1)}{2} + \frac{c_1(M_1)^2}{2}$;*
(d) *$c_1(N).c_1(M/N) \geq \frac{g+3}{2}$.*

Proof. From Proposition 4.1 and Proposition 4.3, we see that $\det M_1 \otimes \mathcal{O}_C$ contributes to $\gamma(C)$.

We have the following four cases:

(i) $h^0(S, M/N), h^0(S, N) \geq 2$
(ii) $h^0(S, M/N) \geq 2$ and $h^0(S, N) < 2$
(iii) $h^0(S, M/N) < 2$ and $h^0(S, N) \geq 2$
(iv) $h^0(S, M/N), h^0(S, N) < 2$

In case (i), we have $h^0(S, H \otimes (M/N)^\vee) = h^0(S, \det M_1 \otimes N) \geq 2$ and $h^0(S, H \otimes N^\vee) = h^0(S, \det M_1 \otimes M/N) \geq 2$ as $\det M_1$ has global sections. Thus we are in case (a) of the lemma.

In case (ii), we see that $\chi(N) < 2$, hence $c_1(N)^2 \leq -2$, and we calculate

$$\begin{aligned} c_1(N).(c_1(M_1).c_1(M/N)) &= c_1(N).(c_1(E) - c_1(N)) \\ &= \mu(N) - c_1(N)^2 \geq \mu(E) + 2 = \frac{g-1}{2} + 2, \end{aligned}$$

thus the first statement of case (b) is proved. We now observe that $c_1(N \otimes \det M_1)^2 > c_1(M/N)^2$ which follows from the computation $c_1(N \otimes \det M_1)^2 - c_1(M/N)^2 \geq 2\mu(M_1) > 0$.

If $c_1(N \otimes \det M_1)^2 < 0$, then also $c_1(M/N)^2 < 0$, and we calculate

$$\begin{aligned} 2g - 2 = c_1(E)^2 &= (c_1(N) + c_1(M/N) + c_1(M_1))^2 \\ &= c_1(N \otimes \det M_1)^2 + 2c_1(N \otimes \det M_1).c_1(M/N) + c_1(M/N)^2 \\ &< 2(c_1(N) + c_1(M_1)).c_1(M/N), \end{aligned}$$

thus $c_1(M/N).(c_1(N) + c_1(M_1)) \geq g$. Else $c_1(N \otimes \det M_1)^2 \geq 0$ and so $h^0(S, H \otimes (M/N)^\vee) = h^0(S, N \otimes \det M_1) \geq 2$ and so M/N contributes to $\gamma(C)$. Thus we are in case (b).

In case (iii), since $\det E/N \cong \det M_1 \otimes M/N$, we have $h^0(S, \det M_1 \otimes M/N) \geq 2$. Thus as $h^0(S, N) \geq 2$, we see that $N \otimes \mathcal{O}_C$ contributes to $\gamma(C)$. Therefore, as $h^0(S, M/N) < 2$, we have $c_1(M/N)^2 \leq -2$.

In cases (iii) and (iv), we have $c_1(M/N)^2 \leq -2$. We now calculate

$$\begin{aligned} 2g - 2 = c_1(E)^2 &= c_1(M/N)^2 + c_1(N)^2 + c_1(M_1)^2 + 2c_1(M/N).c_1(N) + 2c_1(M/N).c_1(M_1) + 2c_1(N).c_1(M_1) \\ &\leq c_1(N)^2 + c_1(M_1)^2 + 2c_1(M/N).c_1(N) + 2c_1(M/N).c_1(M_1) + 2c_1(N).c_1(M_1) - 2 \\ &\leq c_1(N)^2 + g - 3 + 2c_1(M/N).c_1(N), \end{aligned}$$

thus

$$c_1(N).c_1(M/N) \geq \frac{g+1}{2} - \frac{c_1(N)^2}{2}. \tag{10}$$

In case (iii), we observe that since

$$c_1(M/N).(c_1(N) + c_1(M_1)) + c_1(M/N)^2 = \mu(M/N) \geq \mu(E/N) = \frac{(c_1(E/N)).(c_1(E))}{3},$$

we have

$$c_1(M/N).(c_1(N)+c_1(M_1)) \geq -c_1(M/N)^2 + \frac{c_1(M/N)^2}{3} + \frac{c_1(M/N).(c_1(N)+c_1(M_1))}{3} + \frac{c_1(M).c_1(M_1)}{3} + \frac{c_1(M_1)^2}{3}.$$

And subtracting $c_1(M/N).(c_1(N)+c_1(M_1))/3$ from both sides and multiplying by 3/2 yields

$$c_1(M/N).(c_1(N)+c_1(M_1)) \geq -c_1(M/N)^2 + \frac{c_1(M).c_1(M_1)}{2} + \frac{c_1(M_1)^2}{2}.$$

Noting that $c_1(M/N)^2 \leq -2$ shows we are in case (c).

In case (iv), as $h^0(S,N), h^0(S,M/N) < 2$, we have $c_1(N)^2, c_1(M/N)^2 \leq -2$, thus Equation (10) gives $c_1(N).c_1(M/N) \geq \frac{g+1}{2} - \frac{c_1(N)^2}{2} \geq \frac{g+1}{2} + 1 = \frac{g+3}{2}$, and we are in case (d). □

Lemma 4.7. *With E as above, if general curves in $|H|_s$ have Clifford index $\gamma = \gamma(C)$, and $m = D^2$ is the minimum self-intersection of an effective classes $D \in \mathrm{Pic}(S)$ (i.e. there are no curves of genus $g' < \frac{m+2}{2}$ on S), then we have $d \geq 5 + \frac{5}{4}\gamma + \frac{3m}{4}$ or $d \geq 5 + \frac{3}{2}\gamma$.*

Proof. We write $2d \geq 4 + c_1(N).c_1(E/N) + c_1(M/N).(c_1(N)+c_1(M_1)) + c_1(M).c_1(M_1)$. From Proposition 4.1, we see that $c_1(N).c_1(E/N) \geq \gamma + 2$, and $c_1(M).c_1(M_1) \geq \gamma + 2$. In cases (a), (b), we have $c_1(M/N).(c_1(N)+c_1(M_1)) \geq \gamma + 2$. In case (c), we have $d \geq 5 + \frac{5}{4}\gamma + \frac{3m}{4}$. Finally, in case (d), we have $d \geq 2 + c_1(N).c_1(M/N) + c_1(M).c_1(M_1) \geq 2 + \frac{g+13}{2} + \gamma + 2$. And in any case, we have the desired inequality. □

4.4. **Filtration** $1 \subset 3 \subset 4$**.** We assume E has a terminal filtration $0 \subset N \subset M \subset E$ with $\mathrm{rk}(N) = 1$, $\mathrm{rk}(M) = 3$, and M/N a stable torsion free sheaf, and we call $E/M = N_1$. Furthermore, we have

$$\mu(N) \geq \mu(M) \geq \mu(E) \geq \mu(E/N) \geq \mu(N_1) \tag{11}$$

$$\mu(M) \geq \mu(M/N) \geq \mu(E/N) \tag{12}$$

$$d = c_2(E) = c_2(M/N) + c_1(M/N).c_1(N) + c_1(N).c_1(N_1) + c_1(N_1).c_1(M/N) \tag{13}$$

Moreover, since M/N is stable, we have

$$-2 \leq \langle \nu(M/N), \nu(M/N) \rangle = c_1(M/N)^2 - 4\chi(M/N) + 8 = 4c_2(M/N) - c_1(M/N)^2 - 8$$

thus $c_2(M/N) \geq \frac{3}{2} + \frac{c_1(M/N)^2}{4}$.

Lemma 4.8. *Suppose E is as above. Then $N_1 \otimes \mathcal{O}_C$ contributes to $\gamma(C)$, and one of the following occurs:*

(a) *$N \otimes \mathcal{O}_C$ and $\det(M/N) \otimes \mathcal{O}_C$ contribute to $\gamma(C)$;*
(b) *$c_1(N).(c_1(N_1)+c_1(M/N)) \geq \frac{g+3}{2} \geq \gamma(C) + 2$ and either $\det(M/N) \otimes \mathcal{O}_C$ contributes to $\gamma(C)$ or $\frac{c_1(M/N)^2}{2} + c_1(M/N).(c_1(N)+c_1(N_1)) \geq g$;*
(c) *$N \otimes \mathcal{O}_C$ contributes to $\gamma(C)$ and $\frac{c_1(M/N)^2}{2} + c_1(M/N).c_1(N) \geq \frac{1}{2}c_1(N).(c_1(N_1)+c_1(M/N))$;*
(d) *$\frac{c_1(M/N)^2}{2} + c_1(M/N).c_1(N) \geq g+1$.*

Proof. From Proposition 4.1 and Proposition 4.3, we see that $N_1 \otimes \mathcal{O}_C$ contributes to $\gamma(C)$ and $h^2(S, \det M/N) = h^2(S, M/N) = h^2(S,N) = 0$.

We have the following four cases:

(i) $h^0(S, \det M/N), h^0(S,N) \geq 2$
(ii) $h^0(S, \det M/N) \geq 2$ and $h^0(S,N) < 2$
(iii) $h^0(S, \det M/N) < 2$ and $h^0(S,N) \geq 2$

(iv) $h^0(S, \det M/N), h^0(S, N) < 2$

In case (i), we have $h^0(S, H \otimes N^\vee) = h^0(S, \det M/N \otimes N_1) \geq 2$, and $h^0(S, H \otimes \det M/N^\vee) = h^0(S, N \otimes N_1) \geq 2$ as $\det M/N$, N, and N_1 have global sections. Thus we are in case (a) of the lemma.

In case (ii), we see that $\chi(N) < 2$, thus $c_1(N) \leq -2$, and we calculate

$$\begin{aligned} c_1(N).(c_1(N_1) + c_1(M/N)) =& c_1(N).(c_1(E) - c_1(N)) \\ &= \mu(N) - c_1(N)^2 \geq \mu(E) + 2 = \frac{g+3}{2}. \end{aligned}$$

- If $c_1(N \otimes N_1)^2, c_1(M/N)^2 \geq 0$, then $\det M/N$ contributes to $\gamma(C)$ as $h^0(S, N \otimes N_1) = h^0(S, H - \det M/N) \geq 2$.
- If $c_1(N \otimes N_1)^2 \geq 2$ and $c_1(M/N)^2 < 0$, then as above $\det M/N$ contributes to $\gamma(C)$.
- If $c_1(N \otimes N_1)^2 < 0$ and $c_1(M/N)^2 \geq 0$ then we cannot say if $\det M/N$ contributes to $\gamma(C)$ as above. However, we calculate

$$\begin{aligned} 2g - 2 = c_1(E)^2 =& (c_1(M/N) + c_1(N \otimes N_1))^2 \\ &= c_1(M/N)^2 + 2c_1(M/N).(c_1(N) + c_1(N_1)) + c_1(N \otimes N_1)^2 \\ &< c_1(M/N)^2 + 2c_1(M/N).(c_1(N) + c_1(N_1)), \end{aligned}$$

  thus $\frac{c_1(M/N)^2}{2} + c_1(M/N).(c_1(N) + c_1(N_1)) \geq g$.
- If $c_1(N \otimes N_1)^2, c_1(M/N)^2 < 0$, then the same calculation as above yields $\frac{c_1(M/N)^2}{2} + c_1(M/N).(c_1(N) + c_1(N_1)) \geq g$.

Thus we are in case (b) of the lemma.

In case (iii), since $\det E/N = N_1 \otimes \det M/N$, [26, Lemma 3.3] implies that $h^0(S, N_1 \otimes \det M/N) \geq 2$. Thus since $h^0(S, N) \geq 2$, we see that $N \otimes \mathcal{O}_C$ contributes to $\gamma(C)$. Furthermore, as $c_1(M/N)^2 + c_1(M/N).c_1(N) \geq c_1(N_1)^2 + c_1(N_1).c_1(N)$ and $c_1(N_1)^2 \geq 0 > c_1(M/N)^2$, we have $c_1(M/N)^2 + c_1(M/N).c_1(N) \geq c_1(N_1).c_1(N)$. Thus

$$\begin{aligned} c_1(M/N)^2 + c_1(M/N).c_1(N) - \frac{1}{2}(c_1(N).(c_1(N_1) + c_1(M/N))) &\geq c_1(M/N)^2 + \frac{c_1(M/N).c_1(N)}{2} - \frac{c_1(N).c_1(N_1)}{2} \\ &\geq \frac{c_1(M/N)^2}{2}, \end{aligned}$$

thus

$$\frac{c_1(M/N)^2}{2} + c_1(M/N).c_1(N) \geq \frac{1}{2}c_1(N).(c_1(N_1) + c_1(M/N)),$$

and we are in case (c).

In case (iv), we see that $c_1(N)^2, c_1(M/N)^2 \leq -2$. We calculate

$$\begin{aligned} 2g - 2 =& c_1(E)^2 = (c_1(N) + c_1(N_1) + c_1(M/N))^2 \\ &\leq c_1(N_1)^2 + c_1(M/N)^2 + 2c_1(N).c_1(N_1) + 2c_1(N).c_1(M/N) + 2c_1(N_1).c_1(M/N) - 2 \\ &\leq g - 1 + 2c_1(N).c_1(M/N) + c_1(M/N)^2 - 2, \end{aligned}$$

thus $\frac{c_1(M/N)^2}{2} + c_1(N).c_1(M/N) \geq g + 1$, and we are in case (d). □

Remark 4.9. From the second half of the proof of Proposition 4.1, we see that in the situation above, if $C \in |H|_s$ has Clifford index $\gamma = \gamma(C)$, and if $\det M/N$ contributes to $\gamma(C)$, then we have $c_1(M/N).(c_1(N) + c_1(N_1)) \geq \gamma + 2 + 2h^1(S, \det M/N)$.

Lemma 4.10. *With E as above, if general curves in $|H|_s$ have Clifford index $\gamma = \gamma(C)$, we have $d \geq \frac{3}{2}\gamma + 5$.*

Proof. We first see that if $c_1(M/N)^2 \geq 0$, then we are in cases (a) or (b) of the above lemma. Furthermore, we have $c_2(M/N) \geq 2$. Thus in case (a), we have

$$\begin{aligned}
2d &\geq 2c_2(M/N) + c_1(M/N).c_1(N) + c_1(N).c_1(N_1) + c_1(N_1).c_1(M/N) \\
&\geq 4 + 2c_1(M/N).2c_1(N) + 2c_1(N).c_1(N_1) + 2c_1(N_1).c_1(M/N) \\
&= 4 + c_1(M/N).(c_1(N) + c_1(N_1)) + c_1(N).(c_1(N_1) + c_1(M/N)) + c_1(N_1).(c_1(M/N) + c_1(N)) \\
&\geq 4 + 3(\gamma + 2),
\end{aligned}$$

where the last inequality comes from Proposition 4.1. Thus $d \geq \frac{3}{2}\gamma + 5$. In case (b), we calculate as in case (a) and get $d \geq \frac{3}{2}\gamma + 5$ or

$$\begin{aligned}
2d &\geq c_1(N).c_1(N_1) + c_1(N).c_1(M/N) + c_1(N_1).c_1(M/N) + \frac{c_1(M/N)^2}{4} \\
&\geq g + c_1(N).c_1(M/N) + 2c_1(N).c_1(N_1) + c_1(N_1).c_1(M/N) \\
&\geq g + 2(\gamma + 2),
\end{aligned}$$

hence $d \geq \gamma + 2 + \frac{g}{2} > \frac{3}{2}\gamma + 5$.

If $c_1(M/N)^2 < 0$, in case (d), we have

$$\begin{aligned}
d &\geq \frac{3}{2} + \frac{g+1}{2} + \frac{c_1(N).c_1(M/N)}{2} + c_1(M/N).c_1(N_1) + c_1(N).c_1(N_1) \\
&\geq \frac{4+g}{2} + k + \frac{g+1}{2} - \frac{c_1(M/N)^2}{4} \\
&\geq \gamma + 2 + g + \frac{7}{2}.
\end{aligned}$$

If $0 > c_1(M/N)^2 \geq -6$, then $c_2(M/N) \geq 0$, and as there are no (-2)-curves we immediately have that $c_1(M/N)^2 \leq -4$ thus $\chi(\det M/N) \leq 0$. Therefore $h^1(S, \det M/N) \geq h^0(S, \det M/N)$. Calculating as above, we see that

- in case (a), we have $d \geq \frac{3}{2}\gamma + 5$;
- in case (b), we have $d \geq \frac{3}{2}\gamma + 5$ or $d \geq \gamma + \frac{7}{2} + \frac{g+2}{2}$; and,
- in case (c), we have $d \geq \frac{3}{2}\gamma + \frac{3}{2} + 4$.

If $c_2(M/N) < 0$, then the stability of M/N implies that $c_1(M/N)^2 \leq -8$ and

$$-2 \leq \langle \nu(M/N), \nu(M/N) \rangle = c_1(M/N)^2 + 8 - 4\chi(M/N) \leq -4\chi(M/N),$$

whereby $\chi(M/N) \leq 0$. We now consider inequalities associated with various filtrations that lead to the terminal $0 \subset 1 \subset 3 \subset 4$ filtration of E.

If the JH filtration of E is $0 \subset 1 \subset 3 \subset 4$, then we have $p(E) = p(M/N)$, which gives an equality of normalized Euler characteristics

$$\frac{\chi(M/N)}{2} = \frac{\chi(E)}{4} = \frac{g - \gamma + 1}{4}.$$

Thus $0 \geq 2\chi(M/N) = g - d + 7$, and hence $d \geq g + 7$.

If the HN filtration of E is $0 \subset M \subset E$ with $\mathrm{rk}(M) = 3$ and M properly semistable, then the JH filtration of M is $0 \subset N \subset M$. Hence $\mu(M/N) = \mu(M)$ and $\mu(M) > \mu(E)$. Thus

$$\frac{c_1(M/N)^2}{2} + \frac{c_1(M/N).c_1(N \otimes N_1)}{2} = \mu(M/N) > \mu(E) = \frac{g-1}{2},$$

hence

$$\begin{aligned}
d &\geq \frac{3}{2} + \frac{c_1(M/N)^2}{4} + c_1(M/N).c_1(N \otimes N_1) + c_1(N).c_1(N_1) \\
&\geq \frac{3}{2} + \frac{g-1}{2} - \frac{c_1(M/N)^2}{4} + \frac{c_1(M/N).(c_1(N)+c_1(N_1))}{2} \\
&\geq \frac{3}{2} + \frac{g-1}{2} + \frac{c_1(N).(c_1(N_1)+c_1(M/N))}{2} + \frac{c_1(N_1).(c_1(N)+c_1(M/N))}{2} \\
&\geq \frac{3}{2} + \frac{g-1}{2} + \gamma + 1
\end{aligned}$$

where the last inequality comes from the fact that N_1 contributes to $\gamma(C)$, and that in cases (a),(b), and (c) $c_1(N).(c_1(N_1)+c_1(M/N)) \geq \gamma+2$.

If the HN filtration of E is $0 \subset N \subset E$ with E/N properly semistable and the JH filtration of E/N is $0 \subset \overline{M} \subset E/N$ with $\mathrm{rk}(\overline{M}) = 2$, then we have an equality of normalized Euler characteristics

$$\frac{\chi(E)-\chi(N)}{3} = \frac{\chi(E/N)}{3} = \frac{\chi(\overline{M})}{2} = \frac{\chi(M/N)}{2}.$$

Thus $\chi(E) = g-\gamma+1 = \frac{3\chi(M/N)}{2} + \chi(N)$, where $\gamma = d-6$ is the Clifford index of the g^3_d on C. From the short exact sequence

$$0 \longrightarrow N \longrightarrow E \longrightarrow E/N \longrightarrow 0\ ,$$

we have $\chi(N) = h^0(S,E) - h^0(S,E/N) \leq g-\gamma-1$ as $h^0(S,E/N) \geq 2$. Therefore

$$g-\gamma+1 = \chi(E) \leq \frac{3\chi(M/N)}{2} + g-\gamma-1,$$

and thus $2 \leq \frac{3}{2}\chi(M/N) \leq 0$, which is a contradiction. Thus this does not occur, and in all cases we have at least $d \geq \frac{3}{2}\gamma+5$, as claimed. ☐

4.5. **Filtration** $2 \subset 3 \subset 4$. We assume E has a terminal filtration $0 \subset N \subset M \subset E$ with N a stable torsion free sheaf of rank $\mathrm{rk}(N) = 2$, $\mathrm{rk}(M) = 3$, and $N_1 = E/M$ a line bundle. Furthermore, we have

$$\mu(N) \geq \mu(M) \geq \mu(E) = \frac{g-1}{2} \geq \mu(N_1) \tag{14}$$

$$\mu(M) \geq \mu(M/N) \geq \mu(E/N) \geq \mu(N_1) \tag{15}$$

$$d = c_2(E) = c_2(N) + c_1(N).c_1(M/N) + c_1(N).c_1(N_1) + c_1(M/N).c_1(N_1) \tag{16}$$

Moreover, as N is stable, we have $c_2(N) \geq \frac{3}{2} + \frac{c_1(N)^2}{4}$.

Lemma 4.11. *Suppose E is as above. Then $N_1 \otimes \mathcal{O}_C$ contributes to $\gamma(C)$ and one of the following occurs:*

(a) *$(\det N) \otimes \mathcal{O}_C$ and $(M/N) \otimes \mathcal{O}_C$ contribute to $\gamma(C)$;*
(b) *$c_1(N).(c_1(N_1)+c_1(M/N)) \geq g+1$ and either $(M/N) \otimes \mathcal{O}_C$ contributes to $\gamma(C)$ or $c_1(M/N).(c_1(N)+c_1(N_1)) \geq g$;*
(c) *$(\det N) \otimes \mathcal{O}_C$ contributes to $\gamma(C)$, we can assume $c_1(N)^2 \geq 0$ and $c_1(M/N).c_1(N) \geq \frac{1}{2}c_1(N).(c_1(M/N)+c_1(N_1))$;*
(d) *$c_1(N)^2 \leq -2$ and $\frac{c_1(N)^2}{2} + c_1(M/N).c_1(N) \geq \frac{g+1}{2}$.*

Proof. From Proposition 4.1 and Proposition 4.3, we see that $N_1 \otimes \mathcal{O}_C$ contributes to $\gamma(C)$ and $h^2(S,\det N) = h^2(S,\det M) = h^2(S,M/N) = h^2(S,\det E/N) = 0$.

We have the following four cases:

(i) $h^0(S, M/N), h^0(S, \det N) \geq 2$
(ii) $h^0(S, M/N) \geq 2$ and $h^0(S, \det N) < 2$
(iii) $h^0(S, M/N) < 2$ and $h^0(S, \det N) \geq 2$
(iv) $h^0(S, M/N), h^0(S, \det N) < 2$.

In case (i), as N_1 has global sections, and $H - c_1(M/N) = c_1(N) + c_1(N_1)$ and $H - c_1(N) = c_1(N_1) + c_1(M/N)$, we see that both $(\det N) \otimes \mathcal{O}_C$ and $(M/N) \otimes \mathcal{O}_C$ contribute to $\gamma(C)$, and we are in case (a).

In case (ii), we have $\chi(N) < 2$, hence $c_1(N)^2 \leq -2$, and we calculate

$$\begin{aligned} c_1(N).(c_1(N_1) + c_1(M/N)) &= c_1(N).(c_1(E) - c_1(N)) \\ &= 2\mu(N) - c_1(N)^2 \geq g - 1 + 2 = g + 1 \end{aligned}$$

We now observe that $c_1(\det N \otimes N_1)^2 \geq c_1(M/N)^2$ which follows from the following calculation

$$\begin{aligned} c_1(\det N \otimes N_1)^2 - c_1(M/N)^2 &= c_1(N)^2 + 2c_1(N).c_1(N_1) + c_1(N_1)^2 - c_1(M/N)^2 \\ &= 2\mu(N) + \mu(N_1) - \mu(M/N) \geq \mu(N) + \mu(N_1) > 0. \end{aligned}$$

If $c_1(\det N \otimes N_1)^2 < 0$, then also $c_1(M/N) < 0$, and we calculate

$$\begin{aligned} 2g - 2 = c_1(E)^2 &= (c_1(N) + c_1(M/N) + c_1(N_1))^2 \\ &= c_1(\det N \otimes N_1)^2 + 2c_1(\det N \otimes N_1).c_1(M/N) + c_1(M/N)^2 \\ &< 2(c_1(N) + c_1(N_1)).c_1(M/N), \end{aligned}$$

thus $c_1(M/N).(c_1(N) + c_1(N_1)) \geq g$. Else $c_1(\det N \otimes N_1)^2 \geq 0$, and so $h^0(S, H \otimes (M/N)^\vee) = h^0(S, \det N \otimes N_1) \geq 2$, whereby $M/N \otimes \mathcal{O}_C$ contributes to $\gamma(C)$. Thus we are in case (b).

In case (iii), since $\det E/N \cong \det M/N \otimes N_1$, we have $h^0(S, \det M/N \otimes N_1) \geq 2$ by [26, Lemma 3.3]. Thus as $h^0(S, \det N) \geq 2$, we have that $\det N \otimes \mathcal{O}_C$ contributes to $\gamma(C)$. Thus proving the first statement of case (c).

In cases (iii) and (iv), as $h^0(S, M/N) < 2$ we have $c_1(M/N)^2 \leq -2$. We now calculate

$$\begin{aligned} 2g - 2 = c_1(E)^2 &= (c_1(N) + c_1(M/N) + c_1(N_1))^2 \\ &= c_1(N)^2 + c_1(M/N)^2 + c_1(N_1)^2 + 2c_1(N).c_1(M/N) + 2c_1(M/N).c_1(N_1) + 2c_1(N).c_1(N_1) \\ &\leq c_1(N)^2 + c_1(N_1)^2 + 2c_1(N).c_1(M/N) + 2c_1(M/N).c_1(N_1) + 2c_1(N).c_1(N_1) - 2 \\ &\leq g - 1 + c_1(N)^2 + 2c_1(M/N).c_1(N) - 2, \end{aligned}$$

thus $\frac{c_1(N)^2}{2} + c_1(M/N).c_1(N) \geq \frac{g+1}{2}$. If $c_1(N)^2 \leq -2$, we are in case (d).

From now on, we assume $c_1(N)^2 \geq 0$. From the inequality $\mu(M/N) \geq \mu(N_1)$, we see that $c_1(N).c_1(M/N) > c_1(M/N)^2 + c_1(N).c_1(M/N) \geq c_1(N_1)^2 + c_1(N_1).c_1(M/N) \geq c_1(N_1).c_1(M/N)$. Thus

$$c_1(M/N).c_1(N) - \frac{1}{2}c_1(N).(c_1(M/N) + c_1(N_1)) = \frac{1}{2}(c_1(N).c_1(M/N) - c_1(N).c_1(N_1)) > 0,$$

and we are in case (c).

□

Lemma 4.12. *With E as above, if general curves in $|H|_s$ have Clifford index $\gamma = \gamma(C)$, we have $d \geq 5 + \frac{3}{2}\gamma$.*

Proof. The proof follows the same argument as Lemma 4.7. □

4.6. **Filtration** $1 \subset 2 \subset 3 \subset 4$. We suppose E has a terminal filtration of the form

$$0 = E_0 \subset E_1 \subset E_2 \subset E_3 \subset E_4 = E,$$

where $\mathrm{rk}(E_i) = i$, and E_i/E_{i+1} are torsion free sheaves of rank 1. Furthermore, we have

$$\mu(E_1) \geq \mu(E_2) \geq \mu(E_3) \geq \mu(E) = \frac{g-1}{2} \geq \mu(N_1) \tag{17}$$

$$\mu(E_1) \geq \mu(E_2/E_1) \geq \mu(E_3/E_2) \geq \mu(N_1) \tag{18}$$

$$\mu(E_i/E_j) \geq \mu(N_1) \text{ for } 1 \leq j < i \leq 4 \tag{19}$$

(20)

$$d = c_1(N_1).(c_1(E_1) + c_1(E_2/E_1) + c_1(E_3/E_2)) + c_1(E_1).c_1(E_3/E_2) + c_1(E_2/E_1).c_1(E_3/E_2) + c_1(E_1).c_1(E_2/E_1)$$

Letting $e_i := c_1(E_i/(E_{i-1}))$, be the Chern roots of E, and writing $e_i + e_j := c_1(E_i/E_{i-1} \otimes E_j/E_{j-1})$, we have

$$\begin{aligned} 4d =& e_1(e_2+e_3+e_4) + (e_1+e_2).(e_3+e_4) + (e_1+e_2+e_3).e_4 \\ &+ (e_1+e_4).(e_2+e_3) + (e_1+e_3).(e_2+e_4) + (e_1+e_3+e_4).e_2 + (e_1+e_2+e_4).e_3 \end{aligned}$$

Lemma 4.13. *With E as above, if general curves in $|H|_s$ have Clifford index $\gamma = \gamma(C)$,*

$$m := \min\{D^2 | D \in \mathrm{Pic}(S), D^2 \geq 0,\ D \text{ is effective}\}$$

(i.e. there are no curves of genus $g' < \frac{m+2}{2}$ on S), and

$$\mu = \min\{\mu(D) | D \in \mathrm{Pic}(S), D^2 \geq 0, \mu(D) > 0\},$$

we have $d \geq \frac{5}{4}\gamma + \frac{\mu}{2} + \frac{m}{2} + \frac{9}{2}$.

Proof. From Proposition 4.1 and Proposition 4.3, we see that $\det(E/E_i) \otimes \mathcal{O}_C$ contributes to $\gamma(C)$, and so we have $e_1(e_2+e_3+e_4) \geq \gamma+2$, $(e_1+e_2).(e_3+e_4) \geq \gamma+2$, and $(e_1+e_2+e_3).e_4 \geq \gamma+2$. We also have $h^2(S,F) = 0$ for $F = \det(E_i/E_j)$ and $F = N_1, \det E_i$.

It remains to bound the other four terms.

To bound $(e_2+e_3).(e_1+e_4)$, we note that $\mu(e_2+e_3) \geq \mu + \mu(e_3) \geq \mu + \mu(E/E_2)$, and thus

$$(e_2+e_3)^2 + (e_1+e_4).(e_2+e_3) \geq \mu + \frac{(e_1+e_2).(e_3+e_4)}{2} + \frac{(e_3+e_4)^2}{2} \geq \mu + \frac{\gamma+2}{2} + \frac{(e_3+e_4)^2}{2}.$$

Furthermore, we note that $\mu(e_1+e_4) = \mu(e_1) + \mu(e_4) \geq \frac{g-1}{2}$, whereby

$$(e_1+e_4)^2 + (e_1+e_4).(e_2+e_3) \geq \gamma.$$

Now if $h^0(S, e_1+e_4) < 2$ then by considering the Euler characteristic we have $(e_1+e_4)^2 \leq -2$, and thus $(e_1+e_4).(e_2+e_3) \geq \gamma+2$. If $h^0(S, e_2+e_3) < 2$ then $(e_2+e_3)^2 \leq -2$, and we have $(e_1+e_4).(e_2+e_3) \geq 2+\mu+\frac{\gamma+2}{2}+\frac{(e_3+e_4)^2}{2}$. By assumption, $(e_3+e_4)^2 \geq m$, hence $(e_1+e_4).(e_2+e_3) \geq 3+\mu+\frac{\gamma}{2}+\frac{m}{2}$ as well. Finally, if $h^0(S,e_1+e_4), h^0(S,e_2+e_3) \geq 2$, and thus they contribute to the $\gamma(C)$, and hence by Proposition 4.1 $(e_1+e_4).(e_2+e_3) \geq \gamma+2$. Therefore in either case, we have $(e_1+e_4).(e_2+e_3) \geq 3+\mu+\frac{\gamma}{2}+\frac{m}{2}$.

To bound $(e_1+e_3).(e_2+e_4)$, we note that $\mu(e_1+e_3) \geq \frac{g-1}{2}$, and hence

$$(e_1+e_3)^2 + (e_1+e_3).(e_2+e_4) \geq \frac{g-1}{2}.$$

We also note that $\mu(e_2+e_4) \geq \mu + \mu(E/E_1) \geq \mu + \mu(E/E_2)$, whereby

$$(e_2+e_4)^2 + (e_1+e_3).(e_2+e_4) \geq 1 + \frac{(e_1+e_2).(e_3+e_4)}{2} + \frac{(e_3+e_4)^2}{2} \geq 1 + \frac{\gamma+2}{2} + \frac{(e_3+e_4)^2}{2}.$$

As above, we have $(e_1+e_3).(e_2+e_4) \geq 3+\mu+\frac{\gamma}{2}+\frac{m}{2}$.

To bound $(e_1+e_3+e_4).e_2$, we note that $\mu(e_1+e_3+e_4) \geq \mu(e_1) \geq \frac{g-1}{2}$ and $\mu(e_2) \geq \mu(E/E_1) \geq \mu(E/E_2)$. Following the same argument as above, we have $(e_1+e_3+e_4).e_2 \geq 3+\frac{\gamma}{2}+\frac{m}{2}$.

To bound $(e_1+e_2+e_4).e_3$, we note that $\mu(e_1+e_2+e_4) \geq \mu(e_1) \geq \frac{g-1}{2}$ and $\mu(e_3) \geq \mu(E/E_2)$. Following the same argument as above, we have $(e_1+e_2+e_4).e_3 \geq 3+\frac{\gamma}{2}+\frac{m}{2}$.

Finally, we have that three of the terms in the expression for $4d$ are bounded below by $\gamma+2$, two by $3+\frac{\gamma}{2}+\frac{m}{2}$, and two by $3+\mu+\frac{\gamma}{2}+\frac{m}{2}$. Thus $d \geq \frac{5}{4}\gamma+\frac{\mu}{2}+\frac{m}{2}+\frac{9}{2}$, as desired. □

5. Lifting g^3_ds

As above, (S,H) is a polarized K3 surface of genus g, $C \in |H|$ is a smooth irreducible curve of general Clifford index $\gamma = \lfloor \frac{g-1}{2} \rfloor$, A is a complete basepoint free g^3_d with $\rho(A) < 0$, and $E = E_{C,A}$ the unstable LM bundle. Having attained the needed bounds on $c_2(E)$, we can prove our lifting results.

Theorem 5.1. *Let (S,H) be a polarized K3 surface of genus $g \neq 2,3,4,8$ and $C \in |H|$ a smooth irreducible curve of Clifford index γ. Let*

$$m := \min\{D^2 \mid D \in \mathrm{Pic}(S),\ D^2 \geq 0,\ D \textit{ is effective}\}$$

(i.e. there are no curves of genus $g' < \frac{m+2}{2}$ on S), and

$$\mu = \min\{\mu(D) \mid D \in \mathrm{Pic}(S),\ D^2 \geq 0,\ \mu(D) > 0\}.$$

If

$$d < \min\left\{\frac{5}{4}\gamma+\frac{\mu}{2}+\frac{m}{2}+\frac{9}{2},\ \frac{5}{4}\gamma+5+\frac{3m}{4},\ \frac{3}{2}\gamma+5,\ \frac{\gamma}{2}+\frac{g-1}{2}+4\right\},$$

then there is a line bundle $L \in \mathrm{Pic}(S)$ adapted to $|H|$ such that $|A| \subseteq |L \otimes \mathcal{O}_C|$ and $\gamma(L \otimes \mathcal{O}_C) \leq \gamma(A)$. Moreover, one has $c_1(L).C \leq \frac{3g-3}{2}$.

Proof. The LM bundle E has $c_2(E) = d$. If $g \neq 2,3,4,8$, then $d < \frac{5g+19}{6}$. As

$$d < \min\left\{\frac{5}{4}\gamma+\frac{\mu}{2}+\frac{m}{2}+\frac{9}{2},\ \frac{5}{4}\gamma+5+\frac{3m}{4},\ \frac{3}{2}\gamma+5,\ \frac{\gamma}{2}+\frac{g-1}{2}+4\right\}$$

by assumption, the only terminal filtration of E is of type $0 \subset 1 \subset 4$. Thus by Proposition 3.14, the result follows. □

We remark that Theorem 2 is a special case of Theorem 5.1 since if S has no elliptic curves, then $m \geq 2$ so that $d < \frac{5}{4}\gamma+6$.

6. Maximal Brill–Noether Loci in Genus $14-23$

We identify the maximal Brill–Noether loci in genus 14–19, 22, and 23. Our technique uses a few known results about non-containments of Brill–Noether loci, work by Lelli-Chiesa [26] on lifting of rank 2 linear systems and linear systems computing the Clifford index, and our lifting results for rank 3 linear systems above.

We first prove a few useful lemmas which in effect say that if $\mathrm{Pic}(S) = \langle H, L\rangle$ looks like it is obtained by lifting a g^r_d on C to a line bundle L, then L is in fact a lift of a g^r_d. Moreover, when considering these lifts, we would like the line bundle to be basepoint free, which is true if it is primitive. In particular, our next lemma shows that if a curve C on a K3 surface strictly contains a Brill–Noether special linear system, then it is primitive.

Lemma 6.1. *Let (S,H) be a polarized K3 surface of genus g, $C \in |H|$ a smooth connected curve, and $A \in \mathrm{Pic}(C)$ be a line bundle of type g^r_d. Suppose that $\rho(g,r,d) < 0$ and C contains no Brill–Noether special linear series of Clifford index smaller than A. Then A is primitive.*

Proof. We note that $\gamma(\omega_C \otimes A^\vee) = \gamma(A)$, $\rho(A) = \rho(\omega_C \otimes A^\vee)$, $\gamma(A-P) < \gamma(A)$ when P is a basepoint of A, and $\rho(g,r,d-1) < \rho(g,r,d)$. Suppose A has a basepoint P. Then $A-P$ has strictly smaller Clifford index and is Brill–Noether special. By assumption, C cannot be in the linear series $|A-P|$. Thus A is basepoint free. Likewise, if $\omega_C \otimes A^\vee$ has a basepoint P, then $\omega_C \otimes A^\vee - P$ is Brill–Noether special and has smaller Clifford index, which cannot be the case. □

Lemma 6.2. *Let (S,H) be a polarized K3 surface of genus g in the Noether–Lefschetz divisor $\mathcal{K}^r_{g,d}$, i.e., with $\mathrm{Pic}(S)$ admitting a primitive embedding of the sublattice*

$$\Lambda^r_{g,d} = \begin{array}{c|cc} & H & L \\ \hline H & 2g-2 & d \\ L & d & 2r-2 \end{array}$$

Let $C \in |H|$ be a smooth irreducible curve.

- *(i) If $\mathrm{Pic}(S) = \Lambda^r_{g,d}$ and $2 \le r, d \le g-1$, then L is nef.*
- *(ii) If L and $H-L$ are basepoint free, $r \ge 2$, and $0 < d \le g-1$, then $L \otimes \mathcal{O}_C$ is a g^r_d. (The assumption on basepoint free-ness is achieved if for example S has no (-2)-curves, or can be checked numerically.)*
- *(iii) Suppose that $L \otimes \mathcal{O}_C$ is a g^r_d with $\gamma(r,d) > \lfloor \frac{g-1}{2} \rfloor$ and $\rho(g,r,d) < 0$ and that all lattices obtained by lifting special linear systems of general Clifford index or lower cannot be contained in $\mathrm{Pic}(S)$. Then C has Clifford index $\gamma(C) = \lfloor \frac{g-1}{2} \rfloor$, maximal gonality $\lfloor \frac{g+3}{2} \rfloor$, and Clifford dimension 1.*
- *(iv) If $\mathrm{Pic}(S) = \Lambda^r_{g,d}$ is associated to an expected maximal g^r_d, then the assumption on lattices in (iii) holds.*
- *(v) Suppose that $\gamma(r,d) \le \lfloor \frac{g-1}{2} \rfloor$, $\rho(g,r,d) < 0$, and that all lattices obtained by lifting special linear systems A not of type g^r_d with $\gamma(A) \le \lfloor \frac{g-1}{2} \rfloor$ cannot be contained in $\mathrm{Pic}(S)$. Then $L \otimes \mathcal{O}_C$ is a g^r_d and $\gamma(C) = \gamma(r,d)$.*

Proof. To prove (i) we show that for any (-2)-curve $\Gamma = aH + bL \in \Lambda^r_{g,d}$, we have $\Gamma.L \ge 0$. We note that as Γ is a (-2)-curve, a and b must have opposite sign. We prove (i) in three cases.

First suppose $a > 0$ and $b < 0$. Then as $\Gamma.H \ge 1$ and $a > 0$, we have $b\Gamma.L \le -2$, thus as $b < 0$, $\Gamma.L \ge 0$.

Second, suppose $a < -1$ and $b > 0$. Then since $\Gamma.H \ge 1$, we have $a\Gamma.H \le -2$. Thus $b.\Gamma.L \ge 0$, and since $b > 0$ we must have $\Gamma.L \ge 0$.

Lastly, suppose $a = -1$ and $b > 0$. We see that if $\Gamma.H \ge 2$, then we can follow the same argument as above to see that L is nef. Thus the only remaining case is when $a = -1$ and $\Gamma.H = 1$. We calculate $2g-2 = (H+\Gamma)^2 = (bL)^2 = b^2(2r-2)$, hence $b^2 = \frac{g-1}{r-1} \in \mathbb{Z}$. From $\Gamma.H = 1$, we see $b = \frac{2g-1}{d}$, and plugging this in to $2g-2 = b^2(2r-2)$ yields

$$d^2(g-1) = (2g-1)^2(r-1).$$

Looking modulo $g-1$, we immediately see that $r-1 \equiv 0 \mod g-1$, hence $\frac{r-1}{g-1} \in \mathbb{Z}$, and thus $r = g$, which is a contradiction. Thus L is always nef.

To prove (ii), we note that L is clearly a lift of a $g^{r'}_d$ on C for some $r' \ge 0$. Since $0 < d \le g-1$, we see that $L^2, (H-L)^2 > 0$. Furthermore, since $H.L, H.(H-L) > 0$, both these line bundles are nontrivial and intersect H positively, hence $h^0(S,L), h^0(S,H-L) \ge 2$. By assumption, L and $H-L$ are basepoint free, and thus globally generated. Therefore Corollary 3.11 applies. Thus, as $L^2 = 2r-2$, we see that $L \otimes \mathcal{O}_C$ must be a divisor of type g^r_d. Hence (ii) is proved.

To prove (iii), we note that a $g^1_{d'}$ with $\rho(g,1,d') < 0$ has Clifford index $\gamma(g^1_{d'}) < \lfloor \frac{g-1}{2} \rfloor$. Suppose for contradiction that C has lower than general Clifford index. Then by [27, Theorem 4.2] we would be able to lift some special linear system computing $\gamma(C)$ to a divisor $L' \in \mathrm{Pic}(S)$, and by assumption $\langle H, L' \rangle$ cannot be contained in $\mathrm{Pic}(S)$. Thus C has general Clifford index. The same argument shows that C cannot have a special linear system computing its Clifford index. Thus C has a $g^1_{\lfloor \frac{g+3}{2} \rfloor}$ which computes the Clifford index. Hence C has maximal gonality and Clifford dimension 1.

To prove (iv), we note that if C had any Brill–Noether special $g^{r'}_{d'}$ with $\gamma(g^{r'}_{d'}) \leq \frac{g-1}{2}$, then it has a g^r_d with $\gamma = \frac{g-1}{2}$ or a g^1_d with $\gamma(g^1_d) = \frac{g-1}{2} - 1$. Thus we only need to consider lattices $\Lambda^r_{g,d}$ associated to those g^r_d. The proof is now Proposition 1.6(i). Thus (iv) is proved.

To prove (v), we note again that $L \otimes \mathcal{O}_C$ is a $g^{r'}_d$. If $r' \neq r$, then $\gamma(C) \neq \gamma(r, d)$ and some line bundle A would compute $\gamma(C)$. Thus there would exists some lift of A to a line bundle L', but again the lattice $\langle H, L' \rangle \not\subseteq \mathrm{Pic}(S)$. Hence $r' = r$ and we see that $L \otimes \mathcal{O}_C$ is a g^r_d. Similarly, $\gamma(C) = \gamma(r, d)$. □

Remark 6.3. If $\Lambda^r_{g,d}$ has a (-2)-curve, there are still some ways to check that L and $H - L$ are basepoint free. Namely, if they are both nef, then we can check they are basepoint free by checking if there are any elliptic curves on S. Namely if $N \in \mathrm{Pic}(S)$ is nef and there are no elliptic curves, then N is basepoint free by a well-known result of Saint-Donat. To numerically check if $D \in \mathrm{Pic}(S)$ is nef, one can check whether $D.\Gamma \geq 0$ for any (-2)-curve Γ.

One can also check that $L \otimes \mathcal{O}_C$ is a g^r_d by enumerating all of the degree d $g^{r'}_d$ on C and using Lelli-Chiesa's lifting results to show that $\mathrm{Pic}(S)$ cannot have a lift of a $g^{r'}_d$ for $r' \neq r$.

We can now prove that the maximal Brill–Noether loci in genus 14–19, 22, and 23 are as claimed. The proof in genus 14 only requires the lifting results of Lelli-Chiesa. We write the proof in genus 16 and 23 as the other proofs follow similar arguments.

Theorem 6.4. *In genus* 14*, the maximal Brill–Noether loci are* $\mathcal{M}^1_{14,7}$, $\mathcal{M}^2_{14,11}$, *and* $\mathcal{M}^3_{14,13}$.

Proof. By [28, Proposition 2.1] and the trivial containments among Brill–Noether loci, we see that the Brill–Noether loci that can be maximal are the ones above. It remains to show that there are no containments among these loci. By Proposition 1.5, $\mathcal{M}^1_{14,7} \not\subseteq \mathcal{M}^2_{14,11}$ and $\mathcal{M}^1_{14,7} \not\subseteq \mathcal{M}^3_{14,13}$. By (iii) of Lemma 6.2, we see that there are curves which admit a g^2_{11} or a g^3_{13} and have maximal gonality $\lfloor \frac{14+3}{2} \rfloor = 8$, whereby $\mathcal{M}^2_{14,11} \not\subseteq \mathcal{M}^1_{14,7}$ and $\mathcal{M}^3_{14,13} \not\subseteq \mathcal{M}^1_{14,7}$. Since $\rho(14, 2, 11) = -1$ and $\rho(14, 3, 13) = -2$, and noting that therefore $\mathcal{M}^2_{14,11}$ has codimension 1 and $\mathcal{M}^3_{14,13}$ has codimension at least 2 in $\mathcal{M}_{14}$, we see that $\mathcal{M}^2_{14,11} \not\subseteq \mathcal{M}^3_{14,13}$. Finally, Lelli-Chiesa's lifting of rank 2 line bundles [26] shows that $\mathcal{M}^3_{14,13} \not\subseteq \mathcal{M}^2_{14,11}$. □

In genus 16 and 17, the proofs are slightly complicated by the fact that one cannot expect to always exactly lift a linear system $A \in \mathrm{Pic}(C)$, but by the Donagi–Morrison conjecture, we can at least find a line bundle $N \in \mathrm{Pic}(S)$ such that $|A| \subseteq |N \otimes \mathcal{O}_C|$ with $\gamma(N \otimes \mathcal{O}_C) \leq \gamma(A)$. We provide the proof in genus 16, noting that the genus 17 case is similar.

Theorem 6.5. *The maximal Brill–Noether loci in genus* 16 *are* $\mathcal{M}^1_{16,8}$, $\mathcal{M}^2_{16,12}$, *and* $\mathcal{M}^3_{16,14}$.

Proof. One can check as in Remark 6.3 that for $L \in \Lambda^3_{16,14}$, $L \otimes \mathcal{O}_C$ is in fact a g^3_{14}. We note that there are no (-2)-curves in $\Lambda^2_{15,12}$. Hence Lemma 6.2 applies for $\mathrm{Pic}(S)$ either $\Lambda^2_{16,12}$ or $\Lambda^3_{16,14}$. Thus $\mathcal{M}^2_{16,12} \not\subseteq \mathcal{M}^1_{16,8}$ and $\mathcal{M}^3_{16,14} \not\subseteq \mathcal{M}^1_{16,8}$. Furthermore, we have $\mathcal{M}^1_{16,8} \not\subseteq \mathcal{M}^2_{16,12}$ and $\mathcal{M}^1_{16,8} \not\subseteq \mathcal{M}^3_{16,14}$ from Proposition 1.5. It remains to show that there are curves with a g^2_{12} and not a g^3_{14}, and vice versa. Since $\rho(16, 2, 12) = -2$ and $\rho(16, 3, 14) = -4$, we see that $\mathcal{M}^2_{16,12} \not\subseteq \mathcal{M}^3_{16,14}$.

Finally, suppose that $\mathrm{Pic}(S) = \Lambda^3_{16,14}$, and suppose C has a line bundle A of type g^2_{12}. Then by [26, Theorem 1], there is a Donagi–Morrison lift of A. It can easily be checked that if the Donagi–Morrison lift M is not of type g^3_{14}, then M can not be contained in $\mathrm{Pic}(S)$. Thus M is of type g^3_{14} and $M^2 = 4$. However, by Lemma 3.8, we see that $\gamma(E_{C,A}/N) = 0$, and each of the cases in Lemma 3.7 cannot hold. In case (c), one appeals to [37, Theorem 5.2] which shows that a curve is hyperelliptic only if there is an irreducible curve $B \subset S$ of genus 1 or 2. However, this would yield $B^2 = 0$ or $B^2 = 2$, both of which are too small. Thus there can be no M, and thus C cannot admit a g^2_{12}. Thus $\mathcal{M}^3_{16,14} \nsubseteq \mathcal{M}^2_{16,12}$. □

Remark 6.6. In genus 18, the expected maximal Brill–Noether loci are $\mathcal{M}^1_{18,10}$, $\mathcal{M}^2_{18,13}$, and $\mathcal{M}^3_{18,16}$. To prove these are all maximal one notes that when $\mathrm{Pic}(S) = \Lambda^2_{18,13}$, then in Theorem 5.1, we have that $\mu \geq 2$ and hence the Donagi–Morrison conjecture holds for the g^3_{16}, otherwise the argument is similar to the one above. Similarly for genus 19. In genus 22, while our bounds cannot rule out that a curve in $\mathcal{M}^2_{22,16}$ does not admit a g^3_{19}, [3, Corollary 3.5] shows that the loci $\mathcal{M}^2_{22,16}$ and $\mathcal{M}^3_{22,19}$ are distinct. The lifting results of Lelli-Chiesa for g^2_ds and our own lifting results suffice to prove the conjecture in genus 22.

Finally, we provide a proof in genus 23.

Theorem 6.7. *The maximal Brill–Noether loci in genus* 23 *are* $\mathcal{M}^1_{23,12}$, $\mathcal{M}^2_{23,17}$, $\mathcal{M}^3_{23,20}$, *and* $\mathcal{M}^4_{23,22}$.

Remark 6.8. Before the proof, we note that Farkas proved in [11] that the Brill–Noether loci $\mathcal{M}^1_{23,12}$, $\mathcal{M}^2_{23,17}$, and $\mathcal{M}^3_{23,20}$ are mutually distinct. We provide a different proof.

Proof. As before, by [28, Proposition 2.1] and the trivial containments among Brill–Noether loci, we see that the Brill–Noether loci which can be maximal are the ones above. Once again, Proposition 1.5 shows that $\mathcal{M}^1_{23,12}$ is not contained in any of the other loci. Since $\rho(23, 1, 12) = \rho(23, 2, 17) = \rho(23, 3, 20) = -1$, Eisenbud and Harris [10] show that the corresponding loci are irreducible of codimension 1 in $\mathcal{M}_g$ and that $\mathcal{M}^4_{23,22}$ has codimension ≥ 2, hence the other loci cannot be contained in it. Since there are no (-2)-curves in the Picard lattices of a general K3 surface in $\mathcal{K}^2_{23,17}$, $\mathcal{K}^3_{23,20}$, and $\mathcal{K}^4_{23,22}$, we see by Lemma 6.2 that none of the loci are contained in $\mathcal{M}^1_{23,12}$. One can check that for a very general K3 surface in $\mathcal{M}^4_{23,22}$, the minimal positive self-intersection is 4. Hence by Theorem 5.1, if $C \in |H|$ had a g^3_{20} then by considering the Donagi–Morrison lifts, one finds that the g^3_{20} would be contained in $|L|$ restricted to C. After noting that $\gamma(E/N) = 0$, one then argues as in the proof for genus 16. Thus $\mathcal{M}^4_{23,22} \nsubseteq \mathcal{M}^3_{23,20}$. The lifting results in [26] similarly show that $\mathcal{M}^4_{23,22} \nsubseteq \mathcal{M}^2_{23,17}$ and $\mathcal{M}^3_{23,22} \nsubseteq \mathcal{M}^2_{23,17}$. Since the latter two are codimension 1 and irreducible, they are distinct. Thus all of the Brill–Noether loci are distinct. □

7. Lower Genus

If we make a similar assumption on a g^r_d computing the Clifford index of $C \in |H|$, then we have the following theorem.

Theorem 7.1. *For any* $8 \leq g \leq 13$ *and any positive integers* r, d, r', d' *such that*

- $\rho(g, r, d), \rho(g, r', d') < 0$,
- $\Delta(g, r, d), \Delta(g, r', d') < 0$, *and*
- $2 < \gamma(r', d') \leq \gamma(r, d) \leq \lfloor \frac{g-1}{2} \rfloor$,

there is a polarized K3 surface $(S, H) \in \mathcal{K}^r_{g,d}$ *such that a curve* $C \in |H|$ *admits a* g^r_d *but not a* $g^{r'}_{d'}$. *Thus* $\mathcal{M}^r_{g,d} \nsubseteq \mathcal{M}^{r'}_{g,d'}$.

Remark 7.2. The hypothesis that $g \geq 8$ stems from the result in [36, Theorem 1] that a g^1_d lifts if $g > \frac{1}{4}d^2 + d + 2$

Proof. We let (S, H) be a very general polarized K3 surface of genus g in the Noether–Lefschetz divisor $\mathcal{K}^r_{g,d}$, i.e., such that $\mathrm{Pic}(S)$ is given by the lattice

$$\begin{array}{c|cc} & H & L \\ \hline H & 2g-2 & d \\ L & d & 2r-2 \end{array},$$

and $C \in |H|$ a smooth irreducible curve of genus g. As in Proposition 1.6 (i), no lattices obtained by lifting special linear systems on C can be contained in $\mathrm{Pic}(S)$. By Lemma 6.2 (v) we see that $L \otimes \mathcal{O}_C$ is a g^r_d and $\gamma(C) = \gamma(r, d)$. We suppose for contradiction that C admits a $g^{r'}_{d'}$. We cannot have $\gamma(r', d') < \gamma(r, d)$, as then the g^r_d does not compute the Clifford index of C. Hence $\gamma(r', d') = \gamma(r, d)$. But now [27, Theorem 4.2] shows that we can lift the $g^{r'}_{d'}$ to a line bundle $M \in \mathrm{Pic}(S)$, and by Proposition 1.6 (i) again, we see that $\langle H, M\rangle \nsubseteq \mathrm{Pic}(S)$. Thus C cannot admit a $g^{r'}_{d'}$. ☐

From this and Brill–Noether theory for curves of fixed gonality [34, Theorem 1.1], see Proposition 1.5, we can verify the conjecture in low genus.

Corollary 7.3. *In genus* $9 - 13$*, Conjecture 1 holds, i.e., the expected maximal Brill–Noether loci are maximal.*

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