MAXIMAL BRILL–NOETHER LOCI VIA THE GONALITY STRATIFICATION

ASHER AUEL, RICHARD HABURCAK, AND HANNAH LARSON

ABSTRACT. We study the restriction of Brill–Noether loci to the gonality stratification of the moduli space of curves of fixed genus. As an application, we give new proofs that Brill–Noether loci with $\rho = -1, -2$ have distinct support, and for fixed r give lower bounds on when one direction of the non-containments of the Maximal Brill–Noether Loci Conjecture hold for Brill–Noether loci of rank r linear systems. Using these techniques, we also show that Brill–Noether loci corresponding to rank 2 linear systems are maximal as soon as $g \geq 28$ and prove the Maximal Brill–Noether Loci Conjecture for $g = 20$.

INTRODUCTION

If classical Brill–Noether theory concerns linear systems on general algebraic curves, then *refined Brill–Noether theory* can be viewed as the study of linear systems on special curves. The main theorem of Brill–Noether theory [11, 12] implies that the general smooth projective curve C of genus g admits a nondegenerate morphism $C \to \mathbb{P}^r$ of degree d if and only if the *Brill–Noether number*

$$\rho(g, r, d) := g - (r + 1)(g - d + r)$$

is non-negative. A degree d map $C \to \mathbb{P}^r$ determines a degree d line bundle L on C together with a subspace $V \subset H^0(L)$ of dimension $r + 1$. Such a pair (L, V) is called a g^r_d on C.

The last few years have seen a major advance in a refined Brill–Noether theory for curves of fixed gonality [6, 15, 17, 18, 24]. In particular, the general smooth projective k-gonal curve C of genus g admits a g^r_d if and only if Pflueger's Brill–Noether number

$$\rho_k(g, r, d) := \max_{0 \leq \ell \leq r'} \rho(g, r - \ell, d) - \ell k$$

where $r' := \min\{r, g - d + r - 1\}$, is non-negative. More broadly, one of the main goals of refined Brill–Noether theory is to understand when a "general" curve with a g^r_d admits a g^s_e, where here, "general" should mean a general curve in a suitable component of the Brill–Noether locus

$$\mathcal{M}^r_{g,d} = \{C \in \mathcal{M}_g \ : \ C \text{ admits a } g^r_d\}$$

when $\rho(g, r, d) < 0$. Motivated by conjectures concerning lifting line bundles on curves in K3 surfaces, the first two authors posed a conjecture concerning the containments between Brill–Noether loci. Adding basepoints and removing non-basepoints determines various trivial containments between Brill–Noether loci. Accounting for these, one obtains the notion of the *expected maximal Brill–Noether loci*, see Section 1.2.

Conjecture 1 (Maximal Brill–Noether Loci Conjecture)**.** For any $g \geq 3$, except for $g = 7, 8, 9$, the expected maximal Brill–Noether loci are maximal with respect to containment.

In other words, the conjecture states that for any two expected maximal Brill–Noether loci $\mathcal{M}^r_{g,d}$ and $\mathcal{M}^s_{g,e}$, there exists a curve C of genus g admitting a g^r_d but not a g^s_e, and vice versa.

Using the recently established refined Brill–Noether theory for curves of fixed gonality, one deduces (see [1, Proposition 1.6]) that the expected maximal $\mathcal{M}^1_{g,\lfloor \frac{g+1}{2} \rfloor}$ is not contained in any expected maximal Brill–Noether loci $\mathcal{M}^r_{g,d}$ with $r \geq 2$, and is thus maximal, except when $g = 8$. In this note, we explain how additional non-containments between expected maximal Brill–Noether loci can be obtained by restricting to the k-gonal locus. In particular, we obtain the following.

Theorem 1. Fix $r \geq 2$. For g sufficiently large, there is a non-containment $\mathcal{M}^r_{g,d} \nsubseteq \mathcal{M}^s_{g,e}$ for all expected maximal Brill–Noether loci with $s > r$.

In fact, we provide an explicit bound for g in terms of r in Theorem 4.9.

In [1], the first two authors proved the Maximal Brill–Noether Loci Conjecture for $g \leq 19$ and $g = 22, 23$ using K3 surface techniques. In Section 2.1, we show how the techniques developed to prove Theorem 1 by restricting to the k-gonal locus also allow us deduce the following.

Theorem 2. The Maximal Brill–Noether Loci Conjecture holds for $g = 20$.

Furthermore, we reduce the Maximal Brill–Noether Loci Conjecture in genus 21 to verifying just a single non-containment $\mathcal{M}^3_{21,18} \not\subseteq \mathcal{M}^4_{21,20}$.

Outline. In Section 1, we give background on Brill–Noether loci and Brill–Noether theory of curves of fixed gonality. In Section 2, we study the maximum gonality stratum contained in a Brill–Noether locus, and show how it can be used to prove non-containments of Brill–Noether loci. In Section 3, we prove further non-containments of Brill–Noether loci. In Section 4, we focus on expected maximal Brill–Noether loci, give a new proof that Brill–Noether loci with $-\rho \leq 2$ are distinct, and prove an explicit version of Theorem 1 in Theorem 4.9.

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1. Brill–Noether loci

Throughout this paper, we work exclusively over the complex numbers, but we note that there are analogous results for the Brill–Noether theory for curves of fixed gonality in positive characteristic.

1.1. **Brill–Noether loci.** Brill–Noether theory studies maps of curves C to projective space. A nondegenerate morphism $C \to \mathbb{P}^r$ of degree d is determined by a g^r_d, namely, a point in the space

$$G^r_d(C) := \{(L, V) \ | \ L \in \operatorname{Pic}^d(C), \ V \subseteq H^0(C, L), \ \dim V = r + 1\}.$$

The image of the natural map $G^r_d(C) \to \operatorname{Pic}^d(C)$ is

$$W^r_d(C) := \{L \in \operatorname{Pic}^d(C) \ | \ h^0(C, L) \geq r + 1\}.$$

These spaces can be globalized to moduli spaces $\mathcal{G}^r_d \to \mathcal{M}_g$ and $\mathcal{W}^r_d \to \mathcal{M}_g$ over the moduli space $\mathcal{M}_g$ of smooth curves of genus g, where the fiber above C is $G^r_d(C)$ and $W^r_d(C)$, respectively. The Brill–Noether loci

$$\mathcal{M}^r_{g,d} := \{C \in \mathcal{M}_g \mid C \text{ admits a } g^r_d\}$$

are the images of the corresponding maps $\mathcal{G}^r_d \to \mathcal{M}_g$.

The Brill–Noether–Petri theorem [12, 19] states that for a general curve C of genus g, the variety $W^r_d(C)$ is non-empty exactly when the Brill–Noether number

$$\rho(g, r, d) := g - (r + 1)(g - d + r)$$

is non-negative. Consequently, when $\rho(g, r, d) \geq 0$, we have $\mathcal{M}^r_{g,d} = \mathcal{M}_g$. Meanwhile, when $\rho(g, r, d) < 0$, $\mathcal{M}^r_{g,d}$ is a proper subvariety of $\mathcal{M}_g$, all of whose components have codimension at most $-\rho(g, r, d)$ [25]. It is known that Brill–Noether loci with $-3 \leq \rho(g, r, d) \leq -1$ have codimension exactly $-\rho$, and Brill–Noether loci with $\rho = -1, -2$ are irreducible [2, 8, 25].

The stratification of $\mathcal{M}_g$ by Brill–Noether loci and the interaction of various Brill–Noether loci is useful in the study of the birational geometry of $\mathcal{M}_g$, see [9, 13]. Brill–Noether loci with $\rho = -1$ have been studied by Harris, Mumford, Eisenbud, and Farkas [7, 8, 9, 10, 13], in particular, in the study of the Kodaira dimension of $\mathcal{M}_{23}$. More recently, Choi, Kim, and Kim [3, 4] showed in a series of papers that Brill–Noether divisors have distinct support. Choi and Kim [2] showed that

Brill–Noether loci with $\rho = -2$ are irreducible and are not contained in each other; and further showed that Brill–Noether loci with $\rho = -2$ are not contained in certain Brill–Noether divisors; for new proofs of these non-containments see Theorem 4.3 and Theorem 4.4.

1.2. **Expected maximal Brill–Noether loci.** There are various containments among Brill–Noether loci. In particular, there are *trivial containments* $\mathcal{M}^r_{g,d} \subseteq \mathcal{M}^r_{g,d+1}$ obtained by adding a basepoint to a g^r_d on C; and $\mathcal{M}^r_{g,d} \subseteq \mathcal{M}^{r-1}_{g,d-1}$ when $\rho(g, r-1, d-1) < 0$ by subtracting a non-basepoint, cf. [9, 20]. Modulo these trivial containments, the first two authors [1] defined the *expected maximal Brill–Noether loci* as the $\mathcal{M}^r_{g,d}$ where for fixed $r \geq 1$, with $2r \leq d \leq g-1$, d is maximal such that $\rho(g,r,d) < 0$ and $\rho(g, r-1, d-1) \geq 0$. Note that (after accounting for Serre duality which gives $\mathcal{M}^r_{g,d} = \mathcal{M}^{g-d+r-1}_{g,2g-2-d}$) every Brill–Noether locus with $\rho(g,r,d) < 0$ is contained in at least one expected maximal Brill–Noether locus. They then posed Conjecture 1, which says that the expected maximal Brill–Noether loci should be maximal with respect to containment, except when $g = 7, 8, 9$. (In genus $7, 8$ and 9, there are unexpected containments of Brill–Noether loci coming from projections from points of multiplicity ≥ 2 in genus 7 and 9 [1, Propositions 6.2 and 6.4] or from a trisecant line in genus 8, as shown by Mukai [23, Lemma 3.8].)

Let $\gamma(r,d) := d - 2r$ be the Clifford index.

Lemma 1.1. *If $\mathcal{M}^r_{g,d}$ is an expected maximal Brill–Noether locus, then $1 \leq r \leq \left\lfloor \sqrt{g} - \frac{1}{2} \right\rfloor$. Moreover, for each such r, there is an expected maximal Brill–Noether locus $\mathcal{M}^r_{g,d}$.*

Proof. As observed in [1, Remark 1.2], the maximum $\gamma(r,d)$ such that $\rho(g,r,d) \leq 0$ is $g - 2\sqrt{g} + 1$ which occurs at $r = \sqrt{g} - 1$, the intersection of $\rho(g,r,d) = 0$ with $d = g-1$. Thus we have $\gamma(r,d) \leq g + \lfloor -2\sqrt{g} \rfloor + 1$ for an expected maximal Brill–Noether locus. As the trivial containments both increase γ and either fix r or decrease r by one, the maximum r of an expected maximal Brill–Noether locus occurs when $\gamma = g + \lfloor -2\sqrt{g} \rfloor + 1$, or when $\gamma = g - 2\sqrt{g}$ if g is a square. Noting that $d \leq g-1$, we have $\gamma \leq g - 1 - 2r$. If $\sqrt{g} \notin \mathbb{Z}$, then $r \leq \frac{\lceil 2\sqrt{g} \rceil}{2} - 1$. If $\sqrt{g} \in \mathbb{Z}$, then $r \leq \sqrt{g} - \frac{1}{2}$. As r is an integer, and $\left\lfloor \frac{\lceil 2\sqrt{g} \rceil}{2} \right\rfloor - 1 = \left\lfloor \sqrt{g} - \frac{1}{2} \right\rfloor$, the results follow. Finally, since the curve defined by $\rho(g,r,d) = 0$ in the (r,γ)-plane is monotonically increasing for $1 \leq r \leq \sqrt{g} - 1$, there is one expected maximal Brill–Noether locus for each r satisfying these bounds. □

Once a rank r satisfying the conditions of Lemma 1.1 is fixed, the degree d that makes $\mathcal{M}^r_{g,d}$ expected maximal is uniquely determined: it is the largest d such that $\rho(g,r,d) < 0$, namely

$$d = d_{max}(g,r) := r + \left\lceil \frac{gr}{r+1} \right\rceil - 1. \tag{1}$$

For ease of notation, for each $1 \leq r \leq \left\lfloor \sqrt{g} - \frac{1}{2} \right\rfloor$, we shall write $\mathcal{M}^r_g := \mathcal{M}^r_{g,d_{max}(g,r)}$ for the expected maximal Brill–Noether locus of rank r linear series. In other words, Lemma 1.1 says that the expected maximal Brill–Noether loci in $\mathcal{M}_g$ are precisely the $\mathcal{M}^r_g$ for each $1 \leq r \leq \left\lfloor \sqrt{g} - \frac{1}{2} \right\rfloor$.

1.3. **Brill–Noether theory of curves with fixed gonality.** Recall that the *gonality* of a curve is the minimal k such that C admits a g^1_k. The Brill–Noether locus $\mathcal{M}^1_{g,k}$ is the closure of the locus of k-gonal curves. Because the corresponding Hurwitz space of degree k covers is irreducible, $\mathcal{M}^1_{g,k}$ is irreducible. It therefore makes sense to talk about a general k-gonal curve.

In general, $W^r_d(C)$ can have multiple components of varying dimensions. Pflueger [24] showed that for a general k-gonal curve

$$\dim W^r_d(C) \leq \rho_k(g,r,d) := \max_{\ell \in \{0,\ldots,r'\}} \rho(g, r-\ell, d) - \ell k, \tag{2}$$

where $r' := \min\{r, g-d+r-1\}$. Since $W^r_d(C)$ may not have pure dimension, $\dim W^r_d(C)$ above means the maximum of the dimensions of its components. Subsequently, Jensen and Ranganathan [15] showed that a component of the maximum possible dimension exists.

Theorem 1.2 (Jensen–Ranganathan [15])**.** *If C is a general k-gonal curve, then* $\dim W^r_d(C) = \rho_k(g,r,d)$*. In particular, a general k-gonal curve admits a g^r_d if and only if $\rho_k(g,r,d) \geq 0$.*

The dimensions and enumeration of all components of $W^r_d(C)$ were subsequently determined by the third author and others by studying associated splitting loci [5, 6, 15, 17, 18]. For a summary of these results, see [14]. Our applications to maximal Brill–Noether loci will rely only on the statement in Theorem 1.2.

2. The maximal gonality stratum in a Brill–Noether locus

Throughout the remainder of this paper, we add the assumption that $\rho < 0$ for a Brill–Noether locus. Our main new ingredient is the following invariant of a Brill–Noether locus.

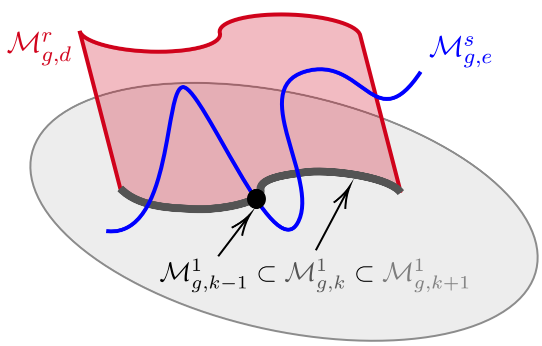
Definition 2.1. For a given genus g, rank r, and degree d, we define $\kappa(g,r,d)$ to be the maximal $k \geq 1$ such that a general curve of genus g and gonality k admits a g^r_d. In other words, $\kappa(g,r,d)$ is the maximal k such that $\mathcal{M}^1_{g,k} \subseteq \mathcal{M}^r_{g,d}$.

Our basic observation is that κ can separate Brill–Noether loci.

Proposition 2.2. *If $\kappa(g,r,d) > \kappa(g,s,e)$, then $\mathcal{M}^r_{g,d} \nsubseteq \mathcal{M}^s_{g,e}$.*

Proof. Assume, to get a contradiction, that $\mathcal{M}^r_{g,d} \subseteq \mathcal{M}^s_{g,e}$, and let $k = \kappa(g,r,d) > \kappa(g,s,e)$. By the definition of κ, $\mathcal{M}^1_{g,k} \nsubseteq \mathcal{M}^s_{g,e}$. But the assumption implies that $\mathcal{M}^1_{g,k} \subseteq \mathcal{M}^r_{g,d} \subseteq \mathcal{M}^s_{g,e}$, which is a contradiction. □

$$\kappa(g,r,d) = k > \kappa(g,s,e) = k-1$$



A general curve of gonality k is contained in $\mathcal{M}^r_{g,d}$, but not in $\mathcal{M}^s_{g,e}$.

Remark 2.3. Noting the trivial containments of Brill–Noether loci, if $\kappa(g,r,d) > \kappa(g,s,e)$ then Proposition 2.2 in fact implies non-containments of Brill–Noether loci of the form $\mathcal{M}^r_{g,d} \nsubseteq \mathcal{M}^s_{g,a}$ for all $a \leq e$ and of the form $\mathcal{M}^r_{g,d} \nsubseteq \mathcal{M}^{s+i}_{g,e+i}$ for all $i \geq 1$.

By Theorem 1.2, a general curve of gonality k admits a g^r_d if and only if $\rho_k(g,r,d) \geq 0$, so

$$\kappa(g,r,d) = \max\{k \;:\; \rho_k(g,r,d) \geq 0\}. \tag{3}$$

We remark that if $\mathcal{M}^r_{g,d}$ is non-empty, then $d - 2r \geq 0$ by Clifford's theorem, from which we can deduce the following bound.

Lemma 2.4. *Let $g,r,d \geq 1$ satisfy $d - 2r \geq d$ and $g - d + r \geq 1$. Then $\kappa(g,r,d) \geq 2$.*

Proof. If we show that $\mathcal{M}^1_{g,2}$ is contained in every non-empty Brill–Noether locus $\mathcal{M}^r_{g,d}$ then it follows that $\kappa(g,r,d) \geq 2$. To this end, if C has a g^1_2, then for any $p \in C$ we have that $rg^1_2 + (d-2r)p$ is a g^r_d on C.

We can also argue directly with Pflueger's formula (2) by setting $\ell = \min\{r, g-d+r-1\}$ and $k = 2$, from which we obtain $\rho_2(g,r,d) \geq d - 2r \geq 0$. Then $\kappa(g,r,d) \geq 2$ by equation (3). □

Despite the combinatorial nature of (2) and (3), we have the following closed formula.

Proposition 2.5. *Suppose $d \le g-1$. We have*

$$\kappa(g,r,d) = \begin{cases} \lfloor \frac{d}{r} \rfloor & \text{if } g+1 > \lfloor \frac{d}{r} \rfloor + d \\ g+1-\gamma(r,d) + \left\lfloor -2\sqrt{-\rho(g,r,d)} \right\rfloor & \text{else.} \end{cases}$$

Proof. Note that $d \le g-1$ is equivalent to $r = \min\{r, g-d+r-1\}$, hence $r = r'$. For fixed g,r,d, we observe that $\rho_k(g,r,d)$ is a non-increasing function of k, as can be seen by writing

$$\rho_k(g,r,d) = \max_{\ell \in \{0,\dots,r\}} \rho(g,r-\ell,d) - \ell k = \max_{\ell \in \{0,\dots,r\}} \rho(g,r,d) + (g-k-\gamma(r,d)+1)\,\ell - \ell^2.$$

In particular, $\rho_k(g,r,d)$ is a maximum over values of a concave down parabola. The maximum of this parabola (over all real values of ℓ) is attained at

$$\ell^* := \frac{g-k-\gamma(r,d)+1}{2}.$$

Thus the maximum of the parabola over our range of integers occurs at $\ell = \lceil \ell^* \rceil$ if $0 \le \ell^* \le r$. Otherwise the maximum of is attained at $\ell = 0$ (if $\ell^* < 0$) or at $\ell = r$ (if $\ell^* > r$).

We now treat each of the two cases in the statement. First suppose $g+1 > \lfloor \frac{d}{r} \rfloor + d$. If $k = \lfloor \frac{d}{r} \rfloor$, then $k < g+1-d$, so $\ell^* > r$ and one checks $\rho_k(g,r,d) = \rho(g,0,d) - rk = d - rk \ge 0$. Meanwhile, if $k = \lfloor \frac{d}{r} \rfloor + 1$, then $k \le g+1-d$, so $\ell^* \ge r$. Hence, $\rho_k(g,r,d) = \rho(g,0,d) - rk = d - rk < 0$. Since $\rho_k(g,r,d)$ is non-increasing, it follows that $\kappa(g,r,d) = \lfloor \frac{d}{r} \rfloor$.

Now suppose $g+1 \le \lfloor \frac{d}{r} \rfloor + d$. In this case, we can bound

$$\begin{aligned} -\rho(g,r,d) &= (r+1)(g-d+r) - g = r(g-d) - d + r^2 + r \\ &\le r\left(\left\lfloor \frac{d}{r} \right\rfloor - 1\right) - d + r^2 + r = r\left\lfloor \frac{d}{r} \right\rfloor - d + r^2 \\ &= d - (d \bmod r) - d + r^2 \le r^2 \end{aligned}$$

where $(d \bmod r)$ denotes the remainder after dividing d by r and where we remark the identity $r\left\lfloor \frac{d}{r} \right\rfloor = d - (d \bmod r)$. Thus $\sqrt{-\rho(g,r,d)} \le r$ and it follows that the claimed value for $\kappa(g,r,d)$ lies in the range $k \ge g+1-d$. If $k \ge g+1-d$, then $\ell^* \le r$, so

$$\rho_k(g,r,d) = \rho(g,r,d) + 2\ell^* \lceil \ell^* \rceil - \lceil \ell^* \rceil^2.$$

If ℓ^* is an integer then $\rho_k(g,r,d) \ge 0$ is equivalent to $(\ell^*)^2 \ge -\rho(g,r,d)$, which in turn is equivalent to

$$k \le g+1-\gamma(r,d) - 2\sqrt{-\rho(g,r,d)}.$$

Otherwise $\lceil \ell^* \rceil = \ell^* + \frac{1}{2}$, so $\rho_k(g,r,d) \ge 0$ is equivalent to

$$\rho(g,r,d) + 2\ell^*(\ell^* + \tfrac{1}{2}) - (\ell^* + \tfrac{1}{2})^2 \ge 0$$

which in turn is equivalent to $(\ell^*)^2 \ge -\rho(g,r,d) - \frac{1}{4}$. In this case, we obtain the bound

$$k \le g+1-\gamma(r,d) - 2\sqrt{-\rho(g,r,d) - \tfrac{1}{4}}.$$

The result now follows from Lemma 2.6 below. □

Lemma 2.6. *For any integer $n > 0$, we have $\lfloor -2\sqrt{n} \rfloor = \left\lfloor -2\sqrt{n - \frac{1}{4}} \right\rfloor$.*

Proof. We see that $\left\lceil 2\sqrt{n-\frac{1}{4}} \right\rceil \le \lceil 2\sqrt{n} \rceil$. Suppose they are not equal. Then there is an $m > 0$ such that $2\sqrt{n-\frac{1}{4}} \le m < 2\sqrt{n}$. Squaring the inequalities gives $4n - 1 \le m^2 < 4n$, whereby $m^2 = 4n-1$. However, since $m^2 \equiv 0, 1 \bmod 4$, we arrive at a contradiction. □

2.1. **Genus 20 and 21.** Using Proposition 2.2, we prove Conjecture 1 in genus 20 and reduce the genus 21 case to a single non-containment.

In genus 20, the expected maximal Brill–Noether loci are $\mathcal{M}^1_{20,10}$, $\mathcal{M}^2_{20,15}$, $\mathcal{M}^3_{20,17}$, $\mathcal{M}^4_{20,19}$.

Theorem 2.7. *The Maximal Brill–Noether Loci Conjecture, Conjecture 1, holds in genus 20.*

Proof. In [1], Conjecture 1 for $g = 20$ was reduced to proving $\mathcal{M}^3_{20,17} \not\subseteq \mathcal{M}^4_{20,19}$. We compute

$$\kappa(20, 3, 17) = 6 > 5 = \kappa(20, 4, 19),$$

whereby Proposition 2.2 gives the desired non-containment. □

In genus 21, the expected maximal Brill–Noether loci are $\mathcal{M}^1_{21,11}$, $\mathcal{M}^2_{21,15}$, $\mathcal{M}^3_{21,18}$, $\mathcal{M}^4_{21,20}$. We summarize the known non-containments without proof, as they follow directly from [1] and Proposition 2.2.

Theorem 2.8. *In genus 21, the loci $\mathcal{M}^1_{21,11}$, $\mathcal{M}^2_{21,15}$, $\mathcal{M}^4_{21,20}$ are maximal. There are also non-containments*

- $\mathcal{M}^3_{21,18} \not\subseteq \mathcal{M}^1_{21,11}$ *and*
- $\mathcal{M}^3_{21,18} \not\subseteq \mathcal{M}^2_{21,15}$.

Remark 2.9. To verify that Conjecture 1 holds in genus 21 the only remaining non-containment is $\mathcal{M}^3_{21,18} \not\subseteq \mathcal{M}^4_{21,20}$

3. Applications to Brill–Noether loci

We apply Proposition 2.2 to prove new non-containments between Brill–Noether loci. We first collect a few observations about ρ and γ for Brill–Noether loci. As κ depends explicitly on γ and ρ, it is natural to ask if γ and ρ are sufficient to numerically identify a g^r_d.

Proposition 3.1. *Let g, r, d, s, e be positive integers. If $\rho(g, r, d) = \rho(g, s, e)$ and $\gamma(r, d) = \gamma(s, e)$, then either*

(i) $r = s$ and $d = e$, or
(ii) $s = g - d + r - 1$ and $e = 2g - 2 - d$.

Proof. Since $\gamma(r, d) = \gamma(s, e)$, writing $s = r + \delta$ gives $e = d + 2\delta$. Simplifying the expression for ρ, we find

$$\rho(g, r, d) = \rho(g, r + \delta, d + 2\delta) = \rho(g, r, d) + \delta(d - g + 1) + \delta^2.$$

Hence we find that either $\delta = 0$ and (i) holds, or $\delta = g - d - 1$ and (ii) holds. □

Remark 3.2. Thus two complete linear systems of type g^r_d and g^s_e are of the same type or of Serre dual type (numerically, $g^s_e = K_C - g^r_d$) if and only if $\rho(g, r, d) = \rho(g, s, e)$ and $\gamma(r, d) = \gamma(s, e)$. In particular, distinct Brill–Noether loci with the same ρ will not have the same γ, and vice versa.

For Brill–Noether loci with the same ρ, Proposition 2.2 easily gives one non-containment. A similar result was recently proved by Teixidor i Bigas in [22].

Corollary 3.3. *Suppose $\rho(g, s, e) = \rho(g, r, d)$ and $g + 1 \leq \left\lfloor \frac{d}{r} \right\rfloor + d, \left\lfloor \frac{e}{s} \right\rfloor + e$. If $\gamma(r, d) < \gamma(s, e)$, then $\mathcal{M}^r_{g,d} \not\subseteq \mathcal{M}^s_{g,e}$.*

As the expected codimension of a Brill–Noether locus $\mathcal{M}^r_{g,d}$ in $\mathcal{M}_g$ is $-\rho(g, r, d)$, one expects non-containments of the form $\mathcal{M}^r_{g,d} \not\subseteq \mathcal{M}^s_{g,e}$ when $\rho(g, r, d) > \rho(g, s, e)$. Thus it is interesting to find non-containments of Brill–Noether loci in the other direction. We give a general statement on when a Brill–Noether locus is not contained in Brill–Noether divisors (loci with $\rho = -1$).

Proposition 3.4. *Suppose that $\rho(g, s, e) = -1$, $e - 2s > d - 2r + \left\lceil 2\sqrt{-\rho(g, r, d)} \right\rceil - 2$, and $g + 1 \leq \left\lfloor \frac{d}{r} \right\rfloor + d$, then $\mathcal{M}^r_{g,d} \not\subseteq \mathcal{M}^s_{g,e}$.*

Proof. Note that if $s = 1$, then $\rho(g, s, e) = -1$ implies $g + 1 \le \left\lfloor \frac{e}{s} \right\rfloor + e$. Simple computations show that when $\rho(g, s, e) = -1$, the condition $\kappa(g, r, d) > \kappa(g, s, e)$ is equivalent to the condition $\gamma(s, e) > \gamma(r, d) + \left\lceil 2\sqrt{-\rho(g, r, d)} \right\rceil - 2$. □

4. Applications to expected maximal Brill–Noether loci

4.1. **Formulas for expected maximal loci.** For expected maximal Brill–Noether loci, one can make the formulas for ρ and κ more explicit. Given g and r with $r \le \sqrt{g} - 1$, recall that we write $d_{max}(g, r)$ for the degree d so that $\mathcal{M}^r_{g,d}$ is expected maximal, given in (1).

Lemma 4.1. *Let $g \bmod r + 1$ be the non-negative representative. For an expected maximal Brill–Noether locus $\mathcal{M}^r_{g,d}$, we have $-\rho(g, r, d) = r + 1 - (g \bmod r + 1)$.*

Proof. We compute

$$\begin{aligned}
\rho(g, r, d_{max}(g, r)) &= g - (r + 1)(g - d_{max}(g, r) + r) \\
&= -gr - r^2 - r + (r + 1)\left(r - 1 + \left\lceil \frac{gr}{r + 1} \right\rceil\right) \\
&= -gr - r - 1 + (r + 1)\left\lceil \frac{gr}{r + 1} \right\rceil.
\end{aligned}$$

Recalling the identity $y \left\lfloor \frac{x}{y} \right\rfloor = x - (x \bmod y)$ for integers x and $y > 0$, we see that

$$(r + 1)\left\lfloor \frac{-gr}{r + 1} \right\rfloor = -gr - (-gr \bmod r + 1) = -gr - (g \bmod r + 1).$$

Thus

$$\begin{aligned}
-\rho(g, r, d_{max}(g, r)) &= r + gr + 1 + (r + 1)\left\lfloor \frac{-gr}{r + 1} \right\rfloor \\
&= r + gr + 1 - gr - (g \bmod r + 1) \\
&= r + 1 - (g \bmod r + 1).
\end{aligned}$$

□

With this formula for $-\rho$, we can simplify our formula for κ for expected maximal loci.

Proposition 4.2. *For an expected maximal Brill–Noether locus $\mathcal{M}^r_{g,d}$ with $r \ge 2$, we have*

$$\kappa(g, r, d) = g + r + 2 + \left\lfloor \frac{-gr}{r + 1} \right\rfloor + \left\lfloor -2\sqrt{r + 1 - (g \bmod (r + 1))} \right\rfloor.$$

Proof. We claim that if $r \ge 2$, and $d = d_{max}(g, r)$, then $g + 1 \le \lfloor \frac{d}{r} \rfloor + d$. Once this is established, combining Lemma 4.1 and Proposition 2.5 and substituting $d = d_{max}(g, r)$ gives the result. To prove the claim, we let $d = d_{max}(g, r) = r - 1 + \lceil \frac{rg}{r+1} \rceil$ and expand

$$\begin{aligned}
\left\lfloor \frac{d}{r} \right\rfloor + d &= \left\lfloor 1 - \frac{1}{r} + \frac{1}{r}\left\lceil \frac{rg}{r + 1} \right\rceil \right\rfloor + r - 1 + \left\lceil \frac{rg}{r + 1} \right\rceil \\
&> r - \frac{1}{r} - 1 + \frac{1}{r}\left\lceil \frac{rg}{r + 1} \right\rceil + \left\lceil \frac{rg}{r + 1} \right\rceil \\
&\ge r - \frac{1}{r} - 1 + g \ge \frac{1}{2} + g.
\end{aligned}$$

Above, we have used that $r - \frac{1}{r}$ is increasing for $r \ge \frac{1}{2}$, so the assumption $r \ge 2$ means $r - \frac{1}{r} \ge 2 - \frac{1}{2}$. Thus, we have $\lfloor \frac{d}{r} \rfloor + d > \frac{1}{2} + g$. Since the left-hand side is an integer, the claim follows. □

4.2. **Non-containments of Brill–Noether loci with $\rho = -1, -2$.** One can easily check that Brill–Noether loci with $\rho = -1, -2$ are expected maximal. Indeed, $\rho(g, r, d+1) = \rho(g, r, d) + r + 1$, and $\rho(g, r-1, d-1) = \rho(g, r, d) + g - d + r$. As shown in [2, 8, 25], Brill–Noether loci with $\rho = -1, -2$ are irreducible. Thus to show that such Brill–Noether loci have distinct support, it suffices to show one non-containment. Choi, Choi, and Kim [2, 4] prove exactly such non-containments. They also give various non-containments of the form $\mathcal{M}^r_{g,d} \nsubseteq \mathcal{M}^s_{g,e}$ when $\rho(g, r, d) = -2$ and $\rho(g, s, e) = -1$. We provide new proofs of these non-containments using κ.

Theorem 4.3. *Let $s \neq r$ and $\rho(g, r, d) = \rho(g, s, e) \in \{-1, -2\}$, then $\mathcal{M}^r_{g,d}$ and $\mathcal{M}^s_{g,e}$ are not contained in each other.*

Proof. If $r = 1$ or $s = 1$, then this follows from [1, Proposition 1.6]. Thus we may assume $r, s \geq 2$. The proof of Proposition 4.2 shows that $g + 1 \leq \left\lfloor \frac{d}{r} \right\rfloor + d, \left\lfloor \frac{s}{e} \right\rfloor + s$. The result now follows from Corollary 3.3. □

Theorem 4.4. *Suppose $\rho(g, s, e) = -1$, $\rho(g, r, d) = -2$, and $e - 2s > d - 2r + 1$, then $\mathcal{M}^r_{g,d} \nsubseteq \mathcal{M}^s_{g,e}$.*

Proof. The case $r = 1$ follows from [1, Proposition 1.6] without the assumption that $e - 2s > d - 2r + 1$. Hence we may assume $r \geq 2$, whereby the proof of Proposition 4.2 implies that $g + 1 \leq \left\lfloor \frac{d}{r} \right\rfloor + d$.

If $s = 1$, then $\rho(g, s, e) = -1$ implies that $g + 1 \leq \left\lfloor \frac{e}{s} \right\rfloor + e$. Thus for any s we have $\kappa(g, s, e) = g + 1 - \gamma(s, e) - 2$. The result now follows from Proposition 3.4. □

Remark 4.5. We note that this slightly improves the bound from [2, Corollary 3.6].

Remark 4.6. In [22, Example 3.2], potential containments of expected maximal Brill–Noether loci of the form $\mathcal{M}^r_{g,d} \subseteq \mathcal{M}^s_{g,e}$ with $\rho(g, r, d) = -2$ and $\rho(g, s, e) = -1$ are given. We briefly recall two such examples, and show that Proposition 3.4 does not address these potential containments.

For the potential containment of the form $\mathcal{M}^{\alpha-1}_{2\alpha^2+\alpha-2, 2\alpha^2-4} \subseteq \mathcal{M}^{\alpha}_{2\alpha^2+\alpha-2, 2\alpha^2-1}$, computing κ shows that the Brill–Noether loci both have $\kappa = 3\alpha - 2$, hence other techniques are required to prove this non-containment.

For the potential containment of the form $\mathcal{M}^{\alpha-1}_{\alpha^2-2, \alpha^2-3} \subseteq \mathcal{M}^{\alpha-2}_{\alpha^2-2, \alpha^2-5}$, computing κ shows that $\kappa(\alpha^2 - 2, \alpha - 1, \alpha^2 - 3) = 2\alpha - 3$, while $\kappa(\alpha^2 - 2, \alpha - 2, \alpha^2 - 5) = 2\alpha - 2$. Thus Proposition 2.2 gives a non-containment $\mathcal{M}^{\alpha-2}_{\alpha^2-2, \alpha^2-5} \nsubseteq \mathcal{M}^{\alpha-1}_{\alpha^2-2, \alpha^2-3}$ (which already follows for dimension reasons), but cannot show non-containment in the other direction.

4.3. **Non-containments of expected maximal Brill–Noether loci.** For the expected maximal loci $\mathcal{M}^r_g$, we have that $r \leq \left\lfloor \sqrt{g} - \frac{1}{2} \right\rfloor$ by Lemma 1.1, and that κ is generally a decreasing function of r. However, it is not strictly decreasing (see Figure 1). Thus, one expects Proposition 2.2 would generally give non-containments $\mathcal{M}^r_g \nsubseteq \mathcal{M}^s_g$ for $r < s$, but to prove such results, we need to control the variation of κ.

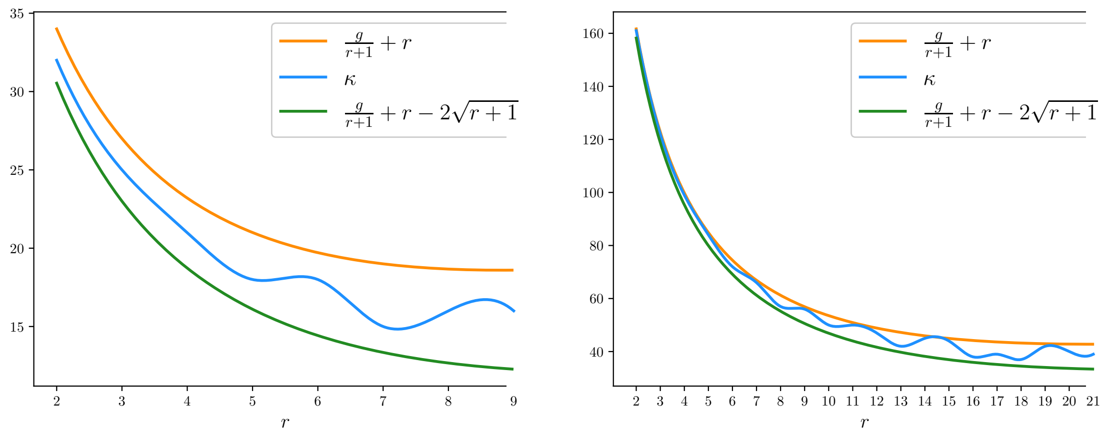
The first step is to give the following bounds on κ, pictured by the orange and green curves in Figure 1.

Lemma 4.7. *For an expected maximal Brill–Noether locus $\mathcal{M}^r_{g,d}$, the following inequalities hold.*

(i) $\kappa(g, r, d) \leq \frac{g}{r+1} + r$.
(ii) $\kappa(g, r, d) > \frac{g}{r+1} + r - 2\sqrt{r+1}$.

Proof. When $r = 1$, we have $\kappa(g, 1, d_{max}(g, 1)) = d_{max}(g, 1) = \left\lceil \frac{g}{2} \right\rceil$, which satisfies the bounds in (i) and (ii). We thus assume $r \geq 2$.

To prove (i), we first observe that since $-\rho \geq 1$, we have $-2\sqrt{-\rho} \leq -2$. We also trivially have $\left\lfloor \frac{-gr}{r+1} \right\rfloor \leq \frac{-gr}{r+1}$, whence (i) follows from Proposition 4.2.



(A) $g = 96$ (B) $g = 479$

FIGURE 1. Plot of $\kappa(g, r, d_{max}(g, r))$.

To prove (ii), we make similar observations. We first note that since $r+1-(g \bmod r+1) \leq r+1$, we have $-2\sqrt{r+1-(g \bmod (r+1))} \geq -2\sqrt{r+1}$, thus

$$\left\lfloor -2\sqrt{r+1-(g \bmod (r+1))} \right\rfloor \geq \left\lfloor -2\sqrt{r+1} \right\rfloor > -2\sqrt{r+1} - 1.$$

Trivially we have $\lfloor \frac{-gr}{r+1} \rfloor > \frac{-gr}{r+1} - 1$, whence (ii) follows from Proposition 4.2. ☐

These bounds give rise to the following criterion for non-containments.

Proposition 4.8. *Let $\delta \geq 1$. If*

$$f(g, r, \delta) := (r+1)\delta^2 + \left((r+1)(r+1+2\sqrt{r+1}) - g\right)\delta + 2(r+1)^2\sqrt{r+1} \leq 0,$$

then $\mathcal{M}^r_g \not\subseteq \mathcal{M}^{r+\delta}_g$.

Proof. The inequality $f(g, r, \delta) \leq 0$ is equivalent to

$$\frac{g}{r+1} + r - 2\sqrt{r+1} \geq \frac{g}{r+\delta+1} + r + \delta.$$

The result then follows from Lemma 4.7 and Proposition 2.2. ☐

Considering $f(g, r, \delta)$ as a quadratic polynomial in δ, we notice that in the limit of large g, the two roots of $f(g, r, \delta)$ tend to 0 and g. Thus, for g sufficiently large, $f(g, r, \delta) \leq 0$ for all $1 \leq \delta \leq \sqrt{g}$. Since expected maximal loci $\mathcal{M}^s_g$ have $s \leq \lfloor \sqrt{g} - \frac{1}{2} \rfloor$ by Lemma 1.1, this implies the non-containment $\mathcal{M}^r_g \not\subseteq \mathcal{M}^s_g$ for all $s > r$. Below we provide an explicit bound on how large g must be in terms of r to achieve all such non-containments.

Theorem 4.9. *Fix $r \geq 2$. If*

$$g \geq 4(r+1)^{5/2} + (r+1)^2 + 2(r+1)^{3/2},$$

then $\mathcal{M}^r_g \not\subseteq \mathcal{M}^s_g$ for all $s > r$.

Proof. Let $\alpha = \sqrt{r+1}$, so that

$$f(g, r, \delta) = \alpha^2\delta^2 + (\alpha^2(\alpha^2 + 2\alpha) - g)\delta + 2\alpha^5.$$

Setting $m = \frac{1}{\alpha^2}(g - \alpha^2(\alpha^2 + 2\alpha))$, we have

$$\frac{1}{\alpha^2} f(g, r, \delta) = \delta^2 - m\delta + 2\alpha^3.$$

Thus, the roots of $f(g, r, \delta)$ are

$$\delta^{\pm} = \frac{1}{2}(m \pm m) \mp \frac{1}{2} m(1 - \sqrt{1 - 8\alpha^3/m^2}).$$

By Proposition 4.8, it suffices to show that $\delta^- \leq 1$ and $\delta^+ \geq \sqrt{g} - 1$. Indeed, if so, then $f(g, r, \delta) \leq 0$ for all $\delta \leq \sqrt{g} - 1$, which implies all desired non-containments.

Note that for $0 \leq x \leq 1$ we have $1 - x \leq \sqrt{1 - x}$, so $1 - \sqrt{1 - x} \leq x$. Thus,

$$\frac{1}{2} m(1 - \sqrt{1 - 8\alpha^3/m^2}) \leq \frac{4\alpha^3}{m}.$$

If $g \geq 4\alpha^5 + \alpha^4 + 2\alpha^3$, then $m \geq 4\alpha^3$. It follows that $\delta^- \leq 1$ and $\delta^+ \geq m - 1$. It thus remains to show that $m \geq \sqrt{g}$, equivalently $m^2 \geq g$, or equivalently

$$g^2 - (3\alpha^4 + 4\alpha^3)g + \alpha^4(\alpha^2 + 2\alpha)^2 \geq 0.$$

The larger root of this quadratic polynomial in g is at

$$\frac{3\alpha^4 + 4\alpha^3 + \sqrt{5\alpha^8 + 8\alpha^7}}{2},$$

which one readily checks is less than $4\alpha^5 + \alpha^4 + 2\alpha^3$ for all $\alpha \geq 1$. □

We have now shown that for each r, there exists a smallest $G(r)$ such that

$$\kappa(g, r, d_{max}(g, r)) > \kappa(g, s, d_{max}(g, s)) \tag{4}$$

for all $g \geq G(r)$ and $r < s \leq \left\lfloor \sqrt{g} - \frac{1}{2} \right\rfloor$. Theorem 4.9 gives an upper bound for $G(r)$, but it is not optimal. Nevertheless, for any fixed r, one can easily check for each of the finitely many $g \leq 4(r+1)^{5/2} + (r+1)^2 + 2(r+1)^{3/2}$ if (4) holds for all $s > r$. We summarize the resulting values of $G(r)$ for low r below.

r	2	3	4	5	6	7	8	9	10
$G(r)$	28	50	96	140	232	306	390	561	684

Remark 4.10. If we fix $r \geq 2$, then there also exist various $g < G(r)$ such that (4) holds for all $s > r$. For example, (4) holds for all $s > r$ when

- $r = 2$ and $g \notin \{10, 11, 12, 15, 18, 19, 24, 27\}$;
- $r = 3$ and $g \notin \{17, 18, 19, 21, 24, 28, 29, 33, 34, 41, 44, 49\}$;
- $r = 4$ and $g \notin \{26, 27, 28, 29, 30, 32, 35, 40, 41, 45, 46, 47, 48, 50, 52, 53, 55, 62, 65, 70, 71, 77, 95\}$.

Corollary 4.11. *Except for $g = 7, 9$, and possibly $g = 24, 27$, the expected maximal Brill–Noether locus $\mathcal{M}^2_g$ is maximal.*

Proof. Write $\mathcal{M}^2_g = \mathcal{M}^2_{g,d}$, so that $d = d_{max}(g, 2)$. The argument for [1, Lemma 6.7 (iii)] shows that for a polarized K3 surface (S, H) of genus $g \geq 14$ with Picard group $\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}L$ with $H.L = d$, and $L^2 = 2$, a smooth curve $C \in |H|$ has general Clifford index, hence general gonality. Indeed, if $\gamma(C) < \left\lfloor \frac{g-1}{2} \right\rfloor$, one first appeals to [16, Lemma 8.3] and [21, Theorem 4.2] and then uses the argument of [1, Lemma 6.7 (iii)]. We claim that C also has a g^2_d, hence $\mathcal{M}^2_g \not\subseteq \mathcal{M}^1_g$. Clearly, $L|_C$ is a g^s_d for some s, and it suffices to show that $s \geq 2$, as then by adding or subtracting points (as in the trivial containments of Brill–Noether loci), C will have a g^2_d.

To this end, remark that since $L.H > 0$, we have $h^2(S, L) = h^0(S, L^\vee) = 0$. Thus applying Riemann–Roch, we see $h^0(S, L) \geq 2 + \frac{L^2}{2} \geq 3$. Note also that $(L - H).H < 0$, so $h^0(S, L - H) = 0$. Now taking the long exact sequence in cohomology associated to the short exact sequence

$$0 \to L \otimes \mathscr{O}_S(-C) \to L \to L \otimes \mathscr{O}_C \to 0,$$

shows that $h^0(C, L|_C) \geq h^0(S, L) \geq 3$, whereby $s \geq 2$, as desired.

Thus $\mathcal{M}^2_g \not\subseteq \mathcal{M}^1_g$ as soon as $g \geq 14$. Likewise, as shown in [1, §6], for $6 \leq g \leq 13$, except for $g = 7, 9$, we also have $\mathcal{M}^2_g \not\subseteq \mathcal{M}^1_g$. As in the above remark, except for possibly $g \in \{10, 11, 12, 15, 18, 19, 24, 27\}$, the expected maximal Brill–Noether locus $\mathcal{M}^2_g$ is not contained in any other expected maximal Brill–Noether locus. From [1], the locus $\mathcal{M}^2_g$ is already known to be maximal when $g = 8, 10, 11, 12, 15, 18, 19$. □

Remark 4.12. More generally, a similar argument involving K3 surfaces with $H.L = d_{max}(g, r)$ and $L^2 = 2r - 2$ shows that $\mathcal{M}^r_g \not\subseteq \mathcal{M}^1_g$ for $g \geq 14$.

Remark 4.13. In case $\kappa(g, r, d_{max}(g, r)) = \kappa(g, s, d_{max}(g, s))$, other techniques are required to prove non-containments. For example, in genus 24, $\kappa(24, 2, 17) = \kappa(24, 4, 23)$. Since $\rho(24, 2, 17) = -3$ and $\rho(24, 4, 23) = -1$, we have a non-containment $\mathcal{M}^4_{24,23} \not\subseteq \mathcal{M}^2_{24,17}$ for dimension reasons. The reverse containment is unknown.

Similarly, in genus 27, $\kappa(27, 2, 19) = \kappa(27, 3, 23)$. As $\rho(27, 2, 19) = -3$ and $\rho(27, 3, 23) = -1$, we have a non-containment $\mathcal{M}^3_{27,23} \not\subseteq \mathcal{M}^2_{27,19}$. The reverse non-containment is unknown.

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Department of Mathematics, Dartmouth College, Kemeny Hall, Hanover, NH 03755
Email address: asher.auel@dartmouth.edu

Email address: richard.haburcak@dartmouth.edu

Department of Mathematics, University of California, Berkeley, Berkeley, CA 94720
Email address: hlarson@berkeley.edu