UNRAMIFIED BRAUER GROUPS OF CONIC BUNDLES OVER RATIONAL SURFACES IN CHARACTERISTIC TWO

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Abstract. We establish a formula for computing the unramified Brauer group of tame conic bundle threefolds in characteristic 2. The formula depends on the arrangement and residue double covers of the discriminant components, the latter being governed by Artin–Schreier theory (instead of Kummer theory in characteristic not 2). We use this to give new examples of threefold conic bundles defined over $\mathbb{Z}$ that are not stably rational over $\mathbb{C}$.

1. Introduction

Motivated by the rationality problem in algebraic geometry, we compute new obstructions to the universal triviality of the Chow group of 0-cycles of smooth projective varieties in characteristic $p > 0$, related to ideas coming from crystalline cohomology (see [CL98] for a survey): the p-torsion in the unramified Brauer group. In [ABBB18], we proved that p-torsion Brauer classes do obstruct the universal triviality of the Chow group of 0-cycles; here we focus on computing these obstructions in the case of conic bundles. In particular, we provide a formula to compute the two torsion in the unramified Brauer group of conic bundle threefolds in characteristic 2. We provide some applications showing that one can obtain results with this type of obstruction that one cannot by other means: there exist conic bundles over $\mathbb{P}^2$, defined over $\mathbb{Z}$, that are smooth over $\mathbb{Q}$ and whose reduction modulo p has (1) nontrivial two torsion in the unramified Brauer group and a universally CH_0-trivial resolution for $p = 2$, and (2) irreducible discriminant, hence trivial Brauer group, for all $p > 2$. The roadmap for this paper is as follows.

In Section 2, we assemble some background on conic bundles and quadratic forms in characteristic 2 to fix notation and the basic notions.

Section 3 contains a few preliminary results about Brauer groups, in particular their p-parts in characteristic p, and then goes on to discuss residue maps, which, in our setting, are only defined on a certain subgroup of the Brauer group, the so-called tamely ramified, or tame, subgroup. We also interpret these residues geometrically for Brauer classes induced by conic bundles in characteristic 2, and show that their vanishing characterizes unramified elements.

Besides the fact that residues are only partially defined, we encounter another new phenomenon in the bad torsion setting, which we discuss in Section 4, namely, that Bloch–Ogus type complexes fail to explain which residue profiles are actually realized by Brauer classes on the base. We also investigate some local analytic normal forms of the discriminants of conic bundles in characteristic 2 that can arise or are excluded for various reasons.

In Section 5 we prove Theorem 5.1, which computes the two torsion in the unramified Brauer group of some conic bundles over surfaces in characteristic two. In the

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hypotheses, we have to assume the existence of certain auxiliary conic bundles over the base with predefined residue subprofiles of the discriminant profile of the initial conic bundle. This is because of the absence of Bloch–Ogus complex methods, as discussed in Section 4.

In Section 6 we construct examples of conic bundle threefolds defined over $\mathbb{Z}$ that are not stably rational over $\mathbb{C}$, of the type described in the first paragraph of this Introduction.

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2. Background on conic bundles

Let K be a field. The most interesting case for us in the sequel will be when K has characteristic 2. Typically, K will not be algebraically closed, for example, the function field of some positive-dimensional algebraic variety over an algebraically closed ground field k of characteristic 2, which is the base space of certain conic bundles or, more generally, quadric fibrations.

2.a. **Quadratic forms.**

As a matter of reference, let us recall here some basic notions concerning the classification of quadratic forms in characteristic 2. We refer to [EKM08, Ch. I–II].

Definition 2.1. Let V be a finite dimensional vector space over K. A *quadratic form* over V is a map $q : V \longrightarrow K$ such that:

a) $q(\lambda v) = \lambda^2 q(v)$ for each $\lambda \in K$ and $v \in V$;

b) the map $b_q : V \times V \longrightarrow K$ defined by

$$b_q(v, w) = q(v + w) - q(v) - q(w)$$

is K-bilinear.

When the characteristic of K is not 2, a quadratic form q can be completely recovered by its associated bilinear form b_q and, thus, by its associated symmetric matrix. This correspondence fails to hold when $\operatorname{char} K = 2$, due to the existence of non-zero quadratic forms with identically zero associated bilinear form; these forms are called *totally singular* and play a significant role in the decomposition of quadratic forms over such fields.

Definition 2.2. Let b be a bilinear form over V; its *radical* is the set

$$r(b) := \{v \in V \mid b(v, w) = 0 \text{ for any } w \in V\}$$

Let q be a quadratic form; the *quadratic radical* is

$$r(q) := \{v \in V \mid q(v) = 0\} \cap r(b_q)$$

In general, we have strict inclusion $r(q) \subset r(b_q)$ if $\operatorname{char} K = 2$. A form such that $r(q) = 0$ is called *regular*.

We introduce the following notation: let q be a quadratic form over V and let $U, W \subseteq V$ be vector subspaces such that $V = W \oplus U$. If U and W are orthogonal with respect to the associated bilinear form b_q (we write $U \subset W^\perp$ to mean this), then q decomposes as sum of its restrictions $q|_W$ and $q|_U$ and we write $q = q|_W \perp q|_V$.

We will also say that two quadratic forms q_1, q_2 defined respectively over V_1 and V_2, are *isometric* if there exists an isometry $f : V_1 \longrightarrow V_2$ of the associated bilinear forms and satisfying $q_1(v) = q_2(f(v))$. In this case, we write $q_1 \simeq q_2$.

Definition 2.3. Let $a, b \in K$. We denote by $\langle a \rangle$ the *diagonal quadratic form* on K (as K-vector space over itself) defined by $v \mapsto a \cdot v^2$. Also, we denote by $[a, b]$ the quadratic form on K^2 defined by $(x, y) \mapsto ax^2 + xy + by^2$.

We say that a quadratic form q is *diagonalizable* if there exists a direct sum decomposition $V = V_1 \oplus \ldots \oplus V_n$ such that each V_i has dimension 1, we have $V_i \subseteq V_j^{\perp}$ for every $i \neq j$ and $q|_{V_i} \simeq \langle a_i \rangle$ so that

$$q \simeq \langle a_1, \ldots, a_n \rangle := \langle a_1 \rangle \perp \ldots \perp \langle a_n \rangle$$

We will also write

$$n \cdot q := \underbrace{q \perp \ldots \perp q}_{n \text{ times}}$$

If $\operatorname{char} K = 2$, then q is diagonalizable if and only if q is totally singular. This is in contrast with the well known case of $\operatorname{char} K \neq 2$, since over such fields every quadratic form is diagonalizable.

A quadratic form q is called *anisotropic* if $q(v) \neq 0$ for every $0 \neq v \in V$. Geometrically, this means that the associated quadric $Q := \{q = 0\} \subset \mathbb{P}(V)$ does not have K-rational points.

A form q is called *non-degenerate* if it is regular and $\dim r(b_q) \leq 1$. Geometrically speaking, non-degeneracy means that the quadric Q is smooth over K, while regularity means that the quadric Q is regular as a scheme, equivalently, is not a cone in $\mathbb{P}(V)$ over a lower dimensional quadric. In characteristic 2, there can exist regular quadratic forms that fail to be non-degenerate.

For example, consider the subvariety X of $\mathbb{P}^2_{(u:v:w)} \times \mathbb{P}^2_{(x:y:z)}$ defined by

$$ux^2 + vy^2 + wz^2 = 0$$

over an algebraically closed field k of characteristic 2. This is a conic fibration over $\mathbb{P}^2_{(u:v:w)}$ such that the generic fiber X_K over $K = k(\mathbb{P}^2_{(u:v:w)})$ is defined by a quadratic form q that is anisotropic, but totally singular. The form q is regular, but fails to be non-degenerate. Geometrically, this means that the conic fibration has no rational section (anisotropic), has a geometric generic fibre that is a double line (totally singular), X_K is not a cone (regular), but X_K is of course not smooth over K. On the other hand, the total space X of this conic fibration is smooth over k.

One has the following structure theorem.

Theorem 2.4. *Let K be a field of characteristic 2 and let q be a quadratic form on a finite-dimensional vector V over K. Then there exist a m-dimensional vector subspace $W \subseteq r(b_q)$ and 2-dimensional vector subspaces $V_1, \ldots, V_s \subseteq V$ such that the following orthogonal decomposition is realized:*

$$q = q|_{r(q)} \perp q|_W \perp q|_{V_1} \perp \ldots \perp q|_{V_s}$$

with $q|_{V_i} \simeq [a_i, b_i]$ for some $a_i, b_i \in K$ a non-degenerate form. Moreover, $q|_W$ is anisotropic, diagonalisable and unique up to isometry. In particular,

$$q \simeq r \cdot \langle 0 \rangle \perp \langle c_1, \ldots, c_m \rangle \perp [a_1, b_1] \perp \ldots \perp [a_s, b_s]$$

We now classify quadratic forms in three variables.

Corollary 2.5. *Let K be a field of characteristic 2, let q be a quadratic forms on a three dimensional vector space V over K, and let $Q \subset \mathbb{P}(V)$ be the associated conic. Then we have the following classification:*

a) *q is non degenerate, equivalently, Q is smooth, and*

$$q \simeq \langle a \rangle \perp [b, c] = ax^2 + by^2 + yz + cz^2$$

where $a \in K^\times$ and $b, c \in K$.

b) *q is regular and totally singular, equivalently, Q is a regular conic that is geometrically a double line, and*

$$q \simeq \langle a, b, c \rangle = ax^2 + by^2 + cz^2$$

where $a, b, c \in K^\times$ and q is anisotropic.

c) $\dim r(b_q) = 1$, *equivalently, Q is a cross of lines over a separable quadratic extension L/K, and*

$$q \simeq \langle 0 \rangle \perp [b, c] = by^2 + yz + cz^2$$

where L is the Artin–Schreier extension of K defined by $x^2 + x + bc$, which is split if and only if Q is a cross of lines over K.

d) $\dim r(q) = 1$ *and q is totally singular, equivalently, Q is a singular conic that is geometrically a double line, and*

$$q \simeq \langle 0, b, c \rangle = by^2 + cz^2$$

where $b, c \in K^\times$.

e) $\dim r(q) = 2$, *equivalently, Q is a double line, and*

$$q \simeq \langle 0, 0, c \rangle = cz^2$$

where $c \in K^\times$.

2.b. **Conic bundles.** Let k be an algebraically closed field. We adopt the following definition of conic bundle.

Definition 2.6. Let X and B be projective varieties over k and let B be smooth. A *conic bundle* is a morphism $\pi : X \longrightarrow B$ such that π is flat and proper with every geometric fibre isomorphic to a plane conic and with smooth geometric generic fibre. In practice, all conic bundles will be given to us in the following form: there is a rank 3 vector bundle $\mathscr{E}$ over B and a quadratic form $q : \mathscr{E} \longrightarrow \mathscr{L}$ (with values in some line bundle $\mathscr{L}$ over B) which is not identically zero on any fibre. Suppose that q is non-degenerate on the generic fibre of $\mathscr{E}$. Then putting $X = \{q = 0\} \subseteq \mathbb{P}(\mathscr{E}) \longrightarrow B$, where the arrow is the canonical projection map to B, defines a conic bundle.

The hypothesis on the geometric generic fibre is not redundant in our context. Suppose that $\operatorname{char} k = 2$, and let $\pi : X \longrightarrow B$ be a flat, proper morphism such that every geometric fibre is isomorphic to a plane conic. Let η be the generic point of B and $K = k(B)$; note that the geometric generic fibre $\overline{X}_\eta$ is a conic in $\mathbb{P}^2_{\overline{K}}$ and it is defined by the vanishing of some quadratic form q_η. By Theorem 2.4, we conclude that then $\overline{X}_\eta$ is cut out by one of the following equations:

$$ax^2 + by^2 + cz^2 = 0 \tag{1}$$

or

$$ax^2 + by^2 + yz + cz^2 = 0 \tag{2}$$

where $(x:y:z)$ are homogeneous coordinates for $\mathbb{P}^2_{\bar{K}}$. The additional assumption on smoothness of the geometric generic fibre allows us to rule out the case of (1), which would give rise to *wild conic bundles*.

We have to define discriminants of conic bundles together with their scheme-structure. First we discuss the discriminant of the generic conic.

Remark 2.7. Let $\mathbb{P}^2$ have homogeneous coordinates $(x:y:z)$ and $\mathbb{P} = \mathbb{P}(H^0(\mathbb{P}^2, \mathscr{O}(2)))$ the 5-dimensional projective space of all conics in $\mathbb{P}^2$. We have the universal conic over $X_{\mathrm{univ}} \longrightarrow \mathbb{P}$ defined as the projection of the incidence $X_{\mathrm{univ}} \subset \mathbb{P} \times \mathbb{P}^2$, which can be written as a hypersurface of bidegree $(1,2)$ defined by the generic conic

$$a_{xx}x^2 + a_{yy}y^2 + a_{zz}z^2 + a_{xy}xy + a_{xz}xz + a_{yz}yz,$$

where we can consider $(a_{xx} : a_{yy} : a_{zz} : a_{xy} : a_{xz} : a_{yz})$ as a system of homogeneous coordinates on $\mathbb{P}$. In these coordinates, the equation of the discriminant $\Delta_{\mathrm{univ}} \subset \mathbb{P}$ parametrizing singular conics is

$$4a_{xx}a_{yy}a_{zz} + a_{xy}a_{yz}a_{xz} - a_{xz}^2a_{yy} - a_{yz}^2a_{xx} - a_{xy}^2a_{zz} \tag{3}$$

which simplifies to

$$a_{xy}a_{yz}a_{xz} + a_{xz}^2a_{yy} + a_{yz}^2a_{xx} + a_{xy}^2a_{zz} \tag{4}$$

in characteristic 2. In any characteristic, $\Delta_{\mathrm{univ}} \subset \mathbb{P}$ is a geometrically integral hypersurface parameterizing the locus of singular conics in $\mathbb{P}^2$.

Definition 2.8. Let $\pi\colon X \to B$ be a conic bundle as in Definition 2.6.

a) The (geometric) discriminant Δ of the conic bundle is the union of those irreducible codimension 1 subvarieties Δ_i in B that have the following property: the geometric generic fibre of the restriction $\pi_{\pi^{-1}(\Delta_i)}\colon X_{\pi^{-1}(\Delta_i)} \to \Delta_i$ is not smooth.

b) We endow Δ with a scheme structure by assigning a multiplicity to each Δ_i as follows. For each i, there is a Zariski open dense subset $U_i \subset B$ such that $\Delta_i \cap U_i \neq \emptyset$ and a morphism $f_i\colon U_i \to \mathbb{P}$ such that $\pi|_{\pi^{-1}(U_i)}\colon X_{\pi^{-1}(U_i)} \to U_i$ is isomorphic to the pull-back via f_i of the universal conic bundle. Then $\Delta_i \cap U_i$ is the reduced subscheme of a component of $f_i^{-1}(\Delta_{\mathrm{univ}})$, interpreted as a scheme-theoretic pullback, and we assign to Δ_i the corresponding multiplicity.

3. Brauer groups and partially defined residues

For a Noetherian scheme X, we denote by $\mathrm{Br}(X)$ Grothendieck's cohomological Brauer group, the torsion subgroup of $H^2_{\text{ét}}(X, \mathbb{G}_{\mathrm{m}})$. If $X = \mathrm{Spec}(A)$ for a commutative ring A, we also write $\mathrm{Br}(A)$ for the Brauer group of $\mathrm{Spec}\, A$.

If X is a regular scheme, every class in $H^2_{\text{ét}}(X, \mathbb{G}_{\mathrm{m}})$ is torsion [Gro68, II, Prop. 1.4]. If X is quasi-projective (over any ring), a result of Gabber [deJ03] says that this group equals the Azumaya algebra Brauer group, defined as the group of Azumaya algebras over X up to Morita equivalence.

Below, unless mentioned otherwise, X will be a smooth projective variety over a field k.

In applications one is frequently only given some singular model of X a priori: it thus becomes desirable, especially since some models of such X can be highly singular and difficult to desingularise explicitly, to determine $\mathrm{Br}(X)$ in terms of data associated to the function field $k(X)$ only. This is really the idea behind unramified invariants as for example in [Bogo87], [CTO], [CT95]. We have an inclusion

$$\mathrm{Br}(X) \subset \mathrm{Br}(k(X))$$

by [Gro68, II, Cor. 1.10], given by pulling back to the generic point of X. One wants to single out the classes inside $\mathrm{Br}(k(X))$ that belong to $\mathrm{Br}(X)$ in valuation-theoretic terms. Since the basic reference [CT95] for this often works under the assumption that the torsion orders of the classes in the Brauer group be coprime to the characteristic of k, we state and prove below a result in the generality we need here, although most of its ingredients are scattered in the available literature.

Basic references for valuation theory are [Z-S76] and [Vac06]. Valuation here without modifiers such as "discrete rank 1" etc. means a general Krull valuation.

Definition 3.1. Let X be a smooth proper variety over a field k and let S be a subset of the set of all Krull valuations of the function field $k(X)$ of X. All the valuations we will consider below will be geometric in the sense that they are assumed to be trivial on the ground field k. For $v \in \mathsf{S}$, we denote by $A_v \subset k(X)$ the valuation ring of v. Then we denote by $\mathrm{Br}_{\mathsf{S}}(k(X)) \subset \mathrm{Br}(k(X))$ the set of all those Brauer classes $\alpha \in \mathrm{Br}(k(X))$ that are in the image of the natural map $\mathrm{Br}(A_v) \to \mathrm{Br}(k(X))$ for all $v \in \mathsf{S}$. Specifically, we will consider the following examples of sets S.

a) The set DISC of discrete rank 1 valuations of $k(X)$ with fraction field $k(X)$.
b) The set DIV of all divisorial valuations of $k(X)$ corresponding to some prime divisor D on a model X' of $k(X)$, where X' is assumed to be generically smooth along D.
c) The set DIV/X of all divisorial valuations of $k(X)$ corresponding to a prime divisor on X.

We denote the corresponding subgroups of $\mathrm{Br}(k(X))$ by

$$\mathrm{Br}_{\mathsf{DISC}}(k(X)),\ \mathrm{Br}_{\mathsf{DIV}}(k(X)),\ \mathrm{Br}_{\mathsf{DIV}/X}(k(X))$$

accordingly. In addition, we define

$$\mathrm{Br}_{\mathsf{LOC}}(k(X))$$

as those classes in $\mathrm{Br}(k(X))$ coming from $\mathrm{Br}(\mathcal{O}_{X,x})$ for every (scheme-)point $x \in X$.

Note the containments $\mathsf{DISC} \supset \mathsf{DIV} \supset \mathsf{DIV}/X$, which are all strict in general: for the first, recall that divisorial valuations are those discrete rank 1 valuations v with the property that the transcendence degree of their residue field is $\dim X - 1$ [Z-S76, Ch. VI, §14, p. 88], [Vac06, §1.4, Ex. 5]; and that there are discrete rank 1 valuations that are not divisorial, for example, so-called analytic arcs [Vac06, Ex. 8(ii)].

Theorem 3.2. *Let X be a smooth projective variety over a field k. Then all of the natural inclusions*

$$\mathrm{Br}(X) \subset \mathrm{Br}_{\mathsf{DISC}}(k(X)) \subset \mathrm{Br}_{\mathsf{DIV}}(k(X)) \subset \mathrm{Br}_{\mathsf{DIV}/X}(k(X))$$

are equalities. In general, if X is smooth and not necessarily proper, then we still have the inclusion $\mathrm{Br}(X) \subset \mathrm{Br}_{\mathsf{DIV}/X}(k(X))$ and this is an equality.

To prove this, we need two preliminary results. The first is a purity result for the cohomological Brauer group of a variety over a field.

Theorem 3.3. *Let V be a smooth k-variety, and let $U \subset V$ be an open subvariety such that $V - U$ has codimension ≥ 2 in V. Then the restriction $\mathrm{Br}(V) \to \mathrm{Br}(U)$ is an isomorphism.*

Proof. The proof starts with a reduction to the case of the punctured spectrum of a strictly Henselian regular local ring of dimension ≥ 2. For torsion prime to the characteristic of k, this follows from the absolute cohomological purity conjecture, whose proof, due to Gabber, appears in [Fuji02] or [ILO14]. For the p primary torsion when k has characteristic p, this follows from [Ga93, Thm. 2.5]. See also [Ga04, Thm. 5] and its proof. Recently, purity for the cohomological Brauer group has been established in complete generality over any scheme [Ces17]. □

The second is a standard Meyer–Vietoris exact sequence for étale cohomology.

Theorem 3.4. *Let V be a scheme. Suppose that $V = U_1 \cup U_2$ is the union of two open subsets. For any sheaf $\mathscr{F}$ of abelian groups in the étale topology on V there is a long exact cohomology sequence*

$$0 \to H^0_{\text{ét}}(V, \mathscr{F}) \to H^0_{\text{ét}}(U_1, \mathscr{F}) \oplus H^0_{\text{ét}}(U_2, \mathscr{F}) \to H^0_{\text{ét}}(U_1 \cap U_2, \mathscr{F})$$
$$\to H^1_{\text{ét}}(V, \mathscr{F}) \to H^1_{\text{ét}}(U_1, \mathscr{F}) \oplus H^1_{\text{ét}}(U_2, \mathscr{F}) \to H^1_{\text{ét}}(U_1 \cap U_2, \mathscr{F}) \to \dots$$

which is functorial in $\mathscr{F}$.

Proof. See [Mi80, Ex. 2.24, p. 110]. □

Proof (of Theorem 3.2). Note that to ensure that one has the inclusion $\mathrm{Br}(X) \subset \mathrm{Br}_{\mathsf{DISC}}(k(X))$ one uses the valuative criterion for properness so that every valuation on $k(X)$ is centered at a point of X. However, the inclusion $\mathrm{Br}(X) \subset \mathrm{Br}_{\mathsf{DIV}/X}(k(X))$ holds regardless of any properness assumptions, because every valuation has a centre on X. Also, the inclusions $\mathrm{Br}_{\mathsf{DISC}}(k(X)) \subset \mathrm{Br}_{\mathsf{DIV}}(k(X)) \subset \mathrm{Br}_{\mathsf{DIV}/X}(k(X))$ come immediately from the definitions, so to prove the theorem, it suffices to verify that every class α in $\mathrm{Br}_{\mathsf{DIV}/X}(k(X))$ belongs to $\mathrm{Br}(X)$.

Any class α in $\mathrm{Br}(k(X))$ can be represented by a class, denoted α_V, in $\mathrm{Br}(V)$ where $V \subset X$ is open with complement a union of prime divisors D_i on X. Moreover, as α is in the image of $\mathrm{Br}(\mathcal{O}_{X,\xi_i})$ for ξ_i the generic point of D_i and all i, we have that there are open subsets U_i of X such that $U_i \cap D_i$ is dense in D_i and there exist classes $\alpha_{U_i} \in \mathrm{Br}(U_i)$ whose images in $\mathrm{Br}(k(X))$ agree with α. By Theorem 3.4, using the cohomological description of the Brauer group, we get that there exist an open subset $U \subset X$ with complement $X \backslash U$ of codimension at least 2 and a class $\alpha_U \in \mathrm{Br}(U)$ inducing α in $\mathrm{Br}(k(X))$. But then Theorem 3.3 shows that α comes from $\mathrm{Br}(X)$. □

Remark 3.5. In the setting of Theorem 3.2, we will agree to denote the group $\mathrm{Br}_{\mathsf{DIV}}(k(X))$ by $\mathrm{Br}_{\mathrm{nr}}(k(X))$ and call this the unramified Brauer group of the function field $k(X)$. We will also use this notation for singular X. According to [Hi17], a resolution of singularities should always exist, but we do not need this result: in all our applications we will produce explicit resolutions $\widetilde{X}$, and then we know $\mathrm{Br}_{\mathrm{nr}}(k(X)) = \mathrm{Br}(\widetilde{X})$.

Next we want to characterize elements in $\mathrm{Br}_{\mathrm{nr}}(k(X))$ in terms of partially defined residues in the sense of Merkurjev as in [GMS03, Appendix A]; this is necessitated by the following circumstance: if one wants to give a formula for the unramified Brauer group of a conic bundle over some smooth projective rational base, for example a smooth projective rational surface, the idea of [CTO] is to produce the nonzero Brauer classes on the total space of the given conic bundle as pull-backs of Brauer classes represented by certain other conic bundles on the base whose residue profiles are a proper subset of the residue profile of the given conic bundle. Hence, one also has to understand the geometric meaning of residues because in the course of this approach it becomes necessary to decide when the residues of two conic bundles along one and the same divisor are equal.

Let K be a field of characteristic p. We denote the subgroup of elements in $\mathrm{Br}(K)$ whose order equals a power of p by $\mathrm{Br}(K)\{p\}$. Let v be a discrete valuation of K,

K_v the completion of K with respect to the absolute value induced by v. Denote the residue field of v by $k(v)$ and by $\overline{K}_v$ an algebraic closure of K_v. One can extend v uniquely from K_v to $\overline{K}_v$, the residue field for that extended valuation on $\overline{K}_v$ will be denoted by $\overline{k(v)}$.

By [Artin67, p. 64–67], unramified subfields of $\overline{K}_v$ correspond to separable subfields of $\overline{k(v)}$, and, in particular, there is a maximal unramified extension, with residue field $k(v)^{\mathrm{s}}$ (separable closure), called the *inertia field*, and denoted by K_v^{nr} or $T = T_v$ (for *Trägheitskörper*). One also has that the Galois group $\mathrm{Gal}(K_v^{\mathrm{nr}}/K_v)$ is isomorphic to $\mathrm{Gal}(k(v))$. Now, recall that by the Galois cohomology characterization, the Brauer group $\mathrm{Br}(K_v)\{p\}$ is isomorphic to $H^2(K_v, (K_v^{\mathrm{s}})^\times)\{p\}$ and there is a natural map

$$H^2(\mathrm{Gal}(K_v^{\mathrm{nr}}/K_v), (K_v^{\mathrm{nr}})^\times)\{p\} \to H^2(K_v, (K_v^{\mathrm{s}})^\times)\{p\} \tag{5}$$

which is injective [GMS03, App. A, Lemma A.6, p. 153].

Definition 3.6. With the above setting, we call the image of (5) the *tame subgroup* or *tamely ramified* subgroup of $\mathrm{Br}(K_v)\{p\}$ associated to v, and denote it by $\mathrm{Br}_{\mathrm{tame},v}(K_v)\{p\}$. We denote its preimage in $\mathrm{Br}(K)\{p\}$ by $\mathrm{Br}_{\mathrm{tame},v}(K)\{p\}$ and likewise call it the tame subgroup of $\mathrm{Br}(K)\{p\}$ associated to v.

Writing again v for the unique extension of v to K_v^{nr} we have a group homomorphism

$$v\colon (K_v^{\mathrm{nr}})^\times \to \mathbb{Z}$$

Definition 3.7. Following [GMS03, App. A] one can define a map as the composition

$$r_v\colon \mathrm{Br}_{\mathrm{tame},v}(K)\{p\} \to H^2(\mathrm{Gal}(K_v^{\mathrm{nr}}/K_v), (K_v^{\mathrm{nr}})^\times)\{p\} \to H^2(k(v), \mathbb{Z})\{p\} \simeq H^1(k(v), \mathbb{Q}/\mathbb{Z})\{p\}.$$

which we call the *residue map* with respect to the valuation v. We will say that the residue of an element in $\mathrm{Br}(K)\{p\}$ with respect to a valuation v is *defined* if that element is in $\mathrm{Br}_{\mathrm{tame},v}(K)\{p\}$.

Remark 3.8. If α is a p-torsion element in $\mathrm{Br}(K)$ for which the residue with respect to a v is defined as in Definition 3.7, then $r_v(\alpha) \in H^1(k(v), \mathbb{Z}/p)$. By Artin–Schreier theory [GS06, Prop. 4.3.10], one has $H^1(k(v), \mathbb{Z}/p) \simeq k(v)/\wp(k(v))$ where $\wp\colon k(v) \to k(v)$, $\wp(x) = x^p - x$, is the Artin–Schreier map. This group classifies pairs consisting of a finite cyclic Galois extensions of $k(v)$ with Galois group $\mathbb{Z}/p$ together with a chosen generator. Indeed, these extensions are obtained by adjoining to $k(v)$ the roots of a polynomial $x^p - x - a$ for some $a \in k(v)$. Here the element a is unique up to the substitution

$$a \mapsto \eta a + (c^p - c)$$

where $\eta \in \mathbb{F}_p^\times$ and $c \in k(v)$, see for example [Artin07, §7.2]. In particular, for $p = 2$, one may also identify $H^1(k(v), \mathbb{Z}/2)$ with $\mathrm{\acute{E}t}_2(k(v))$, isomorphism classes of rank 2 étale algebras over $k(v)$ as in [EKM08, p. 402, Ex. 101.1]; more geometrically, if D is a prime divisor on a smooth algebraic variety over a field k and v_D the corresponding valuation, the residue can be thought of as being given by an étale (i.e., separable) double cover of an open part of D.

Remark 3.9. Keep the notation of Definition 3.7. The tame subgroup

$$\mathrm{Br}_{\mathrm{tame},v}(K_v)\{p\} = H^2(\mathrm{Gal}(K_v^{\mathrm{nr}}/K_v), (K_v^{\mathrm{nr}})^\times)\{p\}$$

of $\mathrm{Br}(K_v)\{p\}$ has a simpler description by [GS06, Thm. 4.4.7, Def. 2.4.9]: it is nothing but the subgroup of elements of order a power of p in the relative Brauer group $\mathrm{Br}(K_v^{\mathrm{nr}}/K_v)$ of Brauer classes in $\mathrm{Br}(K_v)$ that are split by the inertia field K_v^{nr}, in other words, are in the kernel of the natural map

$$\mathrm{Br}(K_v) \to \mathrm{Br}(K_v^{\mathrm{nr}}).$$

To explain the name, one can say that the tame subgroup of $\mathrm{Br}(K_v)\{p\}$ consists of those classes that become trivial in $\mathrm{Br}(V)$, where V is the *maximal tamely ramified extension* of K_v, the ramification field (*Verzweigungskörper*) [Artin67, Ch. 4, §2], because the classes of orders a power of p split by $\mathrm{Br}(T)\{p\}$ coincide with those split by $\mathrm{Br}(V)\{p\}$ (since V is obtained from T by adjoining roots $\sqrt[m]{\pi}$ of a uniformizing element π of K_v of orders m not divisible by p and restriction followed by corestriction is multiplication by the degree of a finite extension). Since in characteristic 0 every extension of K_v is tamely ramified, one can say that in general residues are defined on the subgroup of those classes in $\mathrm{Br}(K_v)\{p\}$ that become trivial on $\mathrm{Br}(V)\{p\}$.

The terminology tame subgroup was suggested to us by Burt Totaro, who also kindly provided other references to the literature. It has the advantage of avoiding the confusing terminology "unramified subgroup", also sometimes used, for elements that can have nontrivial residues. The terminology here is consistent with [TiWa15, §6.2, Prop. 6.63], and one could also have called the tamely ramified subgroup the *inertially split part*, following that source, as the two notions coincide in our context. Our terminology is also consistent with the one in [Ka82, Thm. 3].

Remark 3.10. More generally, given a field F of characteristic $p > 0$, one can define a version of Galois cohomology "with mod p coefficients", following Kato [Ka86] or Merkurjev [GMS03, App. A], [Mer15], in the following way: define

$$H^{n+1}(F, \mathbb{Q}_p/\mathbb{Z}_p(n)) := H^2(F, \mathrm{K}_n(F^{\mathrm{s}}))\{p\} \tag{6}$$

where $\mathrm{K}_n(F^{\mathrm{s}})$ is the n-th Milnor K-group of the separable closure of F, and the cohomology on the right hand side is usual Galois cohomology with coefficients in this Galois module. The coefficients $\mathbb{Q}_p/\mathbb{Z}_p(n)$ on the left hand side are just a symbol here to point out the similarity with the case of characteristics coprime to p, though one can also define them via the logarithmic part of the de Rham–Witt complex, where this symbol has meaning as a coefficients complex.

Given a discrete rank 1 valuation v of F with residue field E, one can define a *tame subgroup* (or *tamely ramified subgroup*)

$$H^{n+1}_{\mathrm{tame},v}(F, \mathbb{Q}_p/\mathbb{Z}_p(n)) \subset H^{n+1}(F, \mathbb{Q}_p/\mathbb{Z}_p(n))$$

in this more general setting in such a way that one recovers the definition given for the Brauer group in Definition 3.7 above: following [GMS03, p. 153] let F_v be the completion, F_v^{nr} its maximal unramified extension, and put

$$H^{n+1}_{\mathrm{tame},v}(F_v, \mathbb{Q}_p/\mathbb{Z}_p(n)) := H^2(\mathrm{Gal}(F_v^{\mathrm{nr}}/F_v), \mathrm{K}_n(F_v^{\mathrm{nr}}))\{p\} \subset H^2(F_v, \mathrm{K}_n(F_v^{\mathrm{sep}}))\{p\}$$

(this is actually a subgroup by [GMS03, Lemma A.6]). Then define the subgroup $H^{n+1}_{\mathrm{tame},v}(F, \mathbb{Q}_p/\mathbb{Z}_p(n))$ as the preimage of $H^{n+1}_{\mathrm{tame},v}(F_v, \mathbb{Q}_p/\mathbb{Z}_p(n))$ under the natural map $H^{n+1}(F, \mathbb{Q}_p/\mathbb{Z}_p(n)) \to H^{n+1}(F_v, \mathbb{Q}_p/\mathbb{Z}_p(n))$. The $\mathrm{Gal}(F_v^{\mathrm{nr}}/F_v)$-equivariant residue map in Milnor K-theory

$$\mathrm{K}_n(F_v^{\mathrm{nr}}) \to K_{n-1}(E^{\mathrm{s}})$$

then induces a residue map, defined only on $H^{n+1}_{\mathrm{tame},v}(F, \mathbb{Q}_p/\mathbb{Z}_p(n))$,

$$r_v \colon H^{n+1}_{\mathrm{tame},v}(F, \mathbb{Q}_p/\mathbb{Z}_p(n)) \to H^n(E, \mathbb{Q}_p/\mathbb{Z}_p(n-1)).$$

Note that, naturally, $\mathrm{Gal}(F_v^{\mathrm{nr}}/F_v) \simeq \mathrm{Gal}(E)$.

We want to describe the relation to logarithmic differentials and restrict to the case of p-torsion for simplicity. Given a discrete rank 1 valuation v of F with residue field E, we have the group

$$H^{n+1}_{\mathrm{tame},v}(F,\mathbb{Z}/p(n)) = H^{n+1}_{\mathrm{tame},v}(F,\mathbb{Q}_p/\mathbb{Z}_p(n))[p]$$

and there is a residue map r_v defined on $H^{n+1}_{\mathrm{tame},v}(F,\mathbb{Z}/p(n))$ as the restriction of the above r_v. We now have the following alternative description

$$H^{n+1}(F,\mathbb{Z}/p(n)) = H^1(F,\Omega^n_{\log}(F^{\mathrm{s}}))$$

where the coefficients $\Omega^n_{\log}(F^{\mathrm{s}})$, denoted $\nu(n)_{F^{\mathrm{s}}}$ in other sources, are defined as the kernel in the exact sequence of Galois modules

$$0 \longrightarrow \Omega^n_{\log}(F^{\mathrm{s}}) \longrightarrow \Omega^n_{F^{\mathrm{s}}} \xrightarrow{\gamma-1} \Omega^n_{F^{\mathrm{s}}}/B^n_{F^{\mathrm{s}}} \longrightarrow 0$$

see [CT99, after Prop. 1.4.2]; here $B^n_{F^{\mathrm{s}}}$ is the subspace of boundaries, the image of the differential $d\colon \Omega^{n-1}_{F^{\mathrm{s}}} \to \Omega^n_{F^{\mathrm{s}}}$, and $\gamma-1$ is a generalization of the Artin–Schreier map defined on generators as

$$\gamma - 1\colon \Omega^n_{F^{\mathrm{s}}} \to \Omega^n_{F^{\mathrm{s}}}/B^n_{F^{\mathrm{s}}}$$
$$x\frac{dy_1}{y_1}\wedge\dots\wedge\frac{dy_n}{y_n} \mapsto (x^p-x)\frac{dy_1}{y_1}\wedge\dots\wedge\frac{dy_n}{y_n} \mod B^n_{F^{\mathrm{s}}}$$

with $x\in F^{\mathrm{s}}$, $y_i \in (F^{\mathrm{s}})^\times$. Now by a result of Kato, Bloch–Kato, Gabber, cf. [CT99, Thm. 3.0], one has

$$\Omega^n_{\log}(F^{\mathrm{s}}) \simeq \mathrm{K}_n(F^{\mathrm{s}})/p$$

or, since by Izhboldin's theorem [Wei13, Thm. 7.8, p. 274] the groups $\mathrm{K}_n(F^{\mathrm{s}})$ have no p-torsion, an exact sequence

$$1 \longrightarrow \mathrm{K}_n(F^{\mathrm{s}}) \xrightarrow{\times p} \mathrm{K}_n(F^{\mathrm{s}}) \longrightarrow \Omega^n_{\log}(F^{\mathrm{s}}) \longrightarrow 1$$

of $\mathrm{Gal}(F)$-modules. The latter shows the equivalence of our two definitions by passing to the long exact sequence in Galois cohomology, taking into account that

$$H^1(F,\mathrm{K}_n(F^{\mathrm{s}}))=0$$

by [Izh91, Lemma 6.6]. One then finds

$$H^1(F,\mathbb{Z}/p(0)) \simeq H^1(F,\mathbb{Z}/p) \simeq F/\wp(F)$$
$$H^2(F,\mathbb{Z}/p(1)) \simeq \mathrm{Br}(F)[p]$$

which brings us back again to Definition 3.7.

We can use the two preceding remarks to prove

Theorem 3.11. *Let k be an algebraically closed field of characteristic 2. Let X and B be projective varieties over k, let B be smooth of dimension ≥ 2 and let $\pi\colon X\to B$ be a conic bundle. Let K be the function field of B, let $\alpha\in \mathrm{Br}(K)[2]$ be the Brauer class determined by the conic bundle, and let D be a prime divisor on B. Suppose that one is in either one of the following two cases:*

a) The geometric generic fibre of

$$\pi|_{\pi^{-1}(D)}\colon X|_{\pi^{-1}(D)} \to D$$

is a smooth conic.

b) The geometric generic fibre of

$$\pi|_{\pi^{-1}(D)} \colon X|_{\pi^{-1}(D)} \to D$$

is isomorphic to two distinct lines in the projective plane and D is a reduced component of the discriminant Δ as in Definition 2.8.

Then in both cases, the residue of α with respect to the divisorial valuation v_D determined by D is defined, and in case a) it is zero, whereas in case b) it is the class of the double cover of D induced by the restriction of the conic bundle π over D, which is étale over an open part of D by the assumption on the type of geometric generic fibre.

We need an auxiliary result before commencing the proof.

Lemma 3.12. *Under all the hypotheses of a) or b) of Theorem 3.11, except possibly the reducedness of D, let $P \in D$ be a point where the fibre X_P is a smooth conic or a cross of lines, respectively. Then we can assume that Zariski locally around P the conic bundle is defined by*

$$ax^2 + by^2 + xz + z^2 = 0$$

with x, y, z fibre coordinates and a, b functions on B, both regular locally around P and b not identically zero.

Proof. Locally around P, the conic bundle is given by an equation

$$a_{xx}x^2 + a_{yy}y^2 + a_{zz}z^2 + a_{xy}xy + a_{xz}xz + a_{yz}yz = 0$$

with the a's regular functions locally around P. Since the fibre X_P is smooth or a cross of lines, we have that one of the coefficients of the mixed terms, without loss of generality a_{xz}, is nonzero in P. Introducing new coordinates by the substitution $x \mapsto (1/a_{xz})x$ (here and in the following we treat x, y, z as well as the a's as dynamical variables to ease notation) one gets the form

$$a_{xx}x^2 + a_{yy}y^2 + a_{zz}z^2 + a_{xy}xy + xz + a_{yz}yz = 0.$$

Now the substitutions $x \mapsto x + a_{yz}y, \quad y \mapsto y, \quad z \mapsto z + a_{xy}y$ transforms this into

$$a_{xx}x^2 + a_{yy}y^2 + a_{zz}z^2 + xz = 0.$$

Now if one of a_{xx} or a_{zz} is nonzero in P, without loss of generality $a_{zz}(P) \neq 0$, then multiplying the equation by a_{zz}^{-1} and subsequently applying the substitution $x \mapsto a_{zz}x$ we obtain the desired normal form

$$a_{xx}x^2 + a_{yy}y^2 + xz + z^2 = 0. \tag{7}$$

But if both a_{xx} and a_{zz} vanish in P, then after applying the substitution $x \mapsto x + z$, we get that $a_{zz}(P) \neq 0$ and proceed as before. □

Proof of Theorem 3.11. Let $v = v_D$ be the divisorial valuation associated to D. We have to check that in both cases of the Theorem, $\alpha \in \mathrm{Br}(K_v^{\mathrm{nr}}/K_v)$, see Remark 3.9, in other words, that α is split by K_v^{nr}. We have the normal form of Lemma 3.12 locally around the generic point of D. The conic bundle is then obviously split by the Galois cover of the base defined by adjoining to $k(B)$ the roots of $T^2 + T + a$ because then the quadratic form in Lemma 3.12 acquires a zero. Moreover, that Galois cover does not ramify in the generic point of D because a has no pole along D. Hence it defines an extension of K_v contained in K_v^{nr}. See also [Artin67, Ex. 1, p. 67].

By formula (4) we find that the discriminant of $ax^2 + by^2 + xz + z^2 = 0$ is given by b, hence by our assumption that D is reduced, we can assume b is a local parameter for D, in case b) of the Theorem, or a unit in the generic point of D in case a). The Brauer class $\alpha \in \mathrm{Br}(k(B))\{2\} = H^2(k(B), \mathbb{Q}_2/\mathbb{Z}_2(1))$ [GMS03, p. 152, formula (2) in Ex. A.3] of the conic bundle defined by the preceding formula is a product of the class $b \in K_1(k(B)) = k(B)^\times/(k(B)^\times)^2$ and the class $a \in H^1(k(B), \mathbb{Q}_2/\mathbb{Z}_2)$:

$$\alpha = b \cup a.$$

Here the cup product is formed as in [GMS03, p. 154, (A.7)]. We now conclude the proof in a number of steps.

Step 1. If π is a local equation for D in $\mathscr{O}_{B,D}$, then a polynomial in a with coefficients in k can only vanish along D if a is congruent modulo π to some element in $k^\times$. If a is not congruent modulo π to an element in k, we consequently have that $k(a) \subset k(B)$ is a subfield of the valuation ring $\mathscr{O}_{B,D}$ of v. By [GMS03, p. 154, sentence before formula (A.8)], the element that a induces in $H^1(k(B), \mathbb{Q}_2/\mathbb{Z}_2)$ is in $H^1_{\mathrm{tame},v}(k(B), \mathbb{Q}_2/\mathbb{Z}_2)$, and then formula (A.8) of loc.cit. implies

$$r_v(b \cup a) = a|_D \in k(D)/\wp(k(D)) = H^1(k(D), \mathbb{Z}/2)$$

in case b) of the Theorem, and $r_v(b \cup a) = 0$ in case a) because the element b is then a unit in the valuation ring of D (alternatively, in case a), the Brauer class of the conic bundle clearly comes from $\mathrm{Br}(A)$, where A is the valuation ring of D, hence the residue is defined and is zero; see also proof of Theorem 3.14 below). Since, in case b), $a|_D$ is precisely the element defining the Artin–Schreier double cover induced by the conic bundle on $D = (b = 0)$, the residue is given by this geometrically defined double cover.

Step 2. If a is congruent modulo π to an element in k, and since k is algebraically closed, we can make a change in the fibre coordinate z so that a is actually a power of π times a unit in $\mathscr{O}_{B,D}$. Since $\dim B \geq 2$, we can find a unit $a' \in \mathscr{O}_{B,D}$ that is not congruent to an element in k modulo π, and write $a = (a - a') + a'$. Now applying Step 1 to $a - a'$ and a' finishes the proof since the product $\cup$ is bilinear and r_v is linear, so $a|_D$, the element defining the Artin–Schreier double cover induced by the conic bundle, is equal to the residue of the conic bundle along D in general. ☐

Lemma 3.13. *Let R be a complete discrete valuation ring with field of fractions K and let K^{nr} be the maximal unramified extension of K, as before. Let R^{nr} be the integral closure of R in K^{nr}. Then $\mathrm{Br}(R) = H^2(\mathrm{Gal}(K^{\mathrm{nr}}/K), (R^{\mathrm{nr}})^\times)$.*

Proof. This is [AB68], proof on p. 289 top, combined with the remark in §3, first sentence of proof of Thm. 3.1. ☐

Theorem 3.14. *Let X be a smooth and projective variety over an algebraically closed field k of characteristic p. Assume $\alpha \in \mathrm{Br}(k(X))\{p\}$ is such that the residue $r_{v_D}(\alpha)$ is defined in the sense of Definition 3.7 and is trivial for all divisorial valuations v_D corresponding to prime divisors D on X. Then $\alpha \in \mathrm{Br}_{\mathrm{nr}}(k(X)) = \mathrm{Br}(X)$.*

If $Z \subset X$ is an irreducible subvariety with local ring $\mathscr{O}_{X,Z}$ and the assumption above is only required to hold for all prime divisors D passing through Z, the class α comes from $\mathrm{Br}(\mathscr{O}_{X,Z})$.

Proof. The equality $\mathrm{Br}_{\mathrm{nr}}(k(X)) = \mathrm{Br}(X)$ follows from Theorem 3.2 taking into account Remark 3.5. We will show that under the assumptions above, we have $\alpha \in \mathrm{Br}_{\mathsf{DIV}/X}(k(X))$, which is enough by Theorem 3.2. Putting $K = k(X)$ and $v = v_D$ and keeping the notation of Definition 3.7 we have an exact sequence

$$H^2(\mathrm{Gal}(K_v^{\mathrm{nr}}/K_v), (A_v^{\mathrm{nr}})^\times)\{p\} \longrightarrow H^2(\mathrm{Gal}(K_v^{\mathrm{nr}}/K_v), (K_v^{\mathrm{nr}})^\times)\{p\} \xrightarrow{r_v} H^1(k(v), \mathbb{Q}/\mathbb{Z})\{p\}$$

resulting from the exact sequence of coefficients $1 \to (A_v^{\mathrm{nr}})^\times \to (K_v^{\mathrm{nr}})^\times \to \mathbb{Z} \to 1$ where $(A_v^{\mathrm{nr}})^\times$ is the valuation ring, inside of K_v^{nr}, of the extension of v to K_v^{nr}. Thus it suffices to show that classes in $\mathrm{Br}_{\mathrm{tame},v}(K)\{p\} \subset \mathrm{Br}(K)\{p\}$ that, under the map

$$\mathrm{Br}_{\mathrm{tame},v}(K)\{p\} \to H^2(\mathrm{Gal}(K_v^{\mathrm{nr}}/K_v), (K_v^{\mathrm{nr}})^\times)\{p\} \subset \mathrm{Br}(K_v)\{p\},$$

land in the image of $H^2(\mathrm{Gal}(K_v^{\mathrm{nr}}/K_v), (A_v^{\mathrm{nr}})^\times)\{p\}$ actually come from $\mathrm{Br}(\mathscr{O}_{X,\xi_D})\{p\}$ where $\mathscr{O}_{X,\xi_D}$ is the local ring of D in $k(X)$. Now, by Lemma 3.13, we have

$$H^2(\mathrm{Gal}(K_v^{\mathrm{nr}}/K_v), (A_v^{\mathrm{nr}})^\times) \simeq \mathrm{Br}(A_v)$$

A class γ in $\mathrm{Br}(K)$ whose image γ_v in $\mathrm{Br}(K_v)$ is contained in $\mathrm{Br}(A_v)$ comes from the valuation ring $A = \mathscr{O}_{X,\xi_D}$ of v in K by Lemma 3.15 below, hence is unramified. □

Lemma 3.15. *Let K be the function field of an algebraic variety and v a discrete rank 1 valuation of K. Let $A \subset K$ be the valuation ring, let K_v be the completion of K with respect to v, and let $A_v \subset K_v$ be the valuation ring of the extension of v to K_v. Then a Brauer class $\alpha \in \mathrm{Br}(K)$ whose image in $\mathrm{Br}(K_v)$ comes from a class $\alpha^\sharp \in \mathrm{Br}(A_v)$ is already in the image of $\mathrm{Br}(A)$.*

Proof. This is a special case of [Ha67, Lemma 4.1.3] or [CTPS12, Lemma 4.1], but we include a proof for completeness.

Suppose the class α is represented by an Azumaya algebra $\mathscr{A}$ over K, and that $\alpha^\sharp$ is represented by an Azumaya algebra $\mathscr{B}$ over A_v. By assumption, $\mathscr{A}$ and $\mathscr{B}$ become Brauer equivalent over K_v, and we can assume that they even become isomorphic over K_v by replacing $\mathscr{A}$ and $\mathscr{B}$ by matrix algebras over them so that they have the same degree. Let $\mathscr{A}_A$ be a maximal A-order of the algebra $\mathscr{A}$ in the sense of Auslander–Goldman [AG60], which means that $\mathscr{A}_A$ is a subring of $\mathscr{A}$ that is finitely generated as an A-module, spans $\mathscr{A}$ over K and is maximal with these properties. We seek to prove that $\mathscr{A}_A$ is Azumaya. Now we know that the base change $(\mathscr{A}_A)_{A_v}$ is a maximal order, but also any Azumaya A_v-algebra is a maximal order, and by [AG60, Prop. 3.5], any two maximal orders over a rank 1 discrete valuation ring are conjugate, so in fact the base change $(\mathscr{A}_A)_{A_v}$ is Azumaya because $\mathscr{B}$ is. But then this implies that $\mathscr{A}_A$ is Azumaya since A_v is faithfully flat over A, so if the canonical algebra homomorphism $\mathscr{A}_A \otimes \mathscr{A}_A \to \mathrm{End}(\mathscr{A}_A)$ becomes an isomorphism over A_v, it is already an isomorphism over A. □

Remark 3.16. Here is a more geometric proof of Theorem 3.14 for the case that D is as in Theorem 3.11, b) and the class α is a 2-torsion class represented by a conic bundle $\pi\colon Y \to X$ itself. In that case, it suffices to show that there exists a birational modification

$$\begin{array}{ccc} Y' & \dashrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array}$$

with $Y' \to X'$ a conic bundle square birational to $Y \to X$, $X' \to X$ an isomorphism over the generic point of D and such that the general fibre of Y' over the strict transform of D on X' is a smooth conic.

We can assume the normal form from Lemma 3.12

$$ax^2 + by^2 = xz + z^2$$

whence in characteristic 2

$$by^2 = ax^2 + xz + z^2.$$

By assumption, the right hand side factors modulo b, that is, there is a function α on $(b = 0)$ such that

$$(\alpha x + z)(1 + \alpha x + z) = a|_{(b=0)}x^2 + xz + z^2.$$

Let α' some extension of α to a neighborhood of $(b = 0)$ in the base. Then the matrix

$$\begin{pmatrix} \alpha' & 1 \\ 1+\alpha' & 1 \end{pmatrix}$$

has determinant 1, and thus applying the coordinate transformation

$$x' = \alpha' x + z$$
$$z' = (1+\alpha')x + z$$

one gets

$$by^2 = x'z' + b(u(x')^2 + vx'z' + w(z')^2).$$

Applying $x' \mapsto bx''$ one gets

$$by^2 = bx''z' + b(ub^2(x'')^2 + vbx''z' + w(z')^2)$$

Outside of $b = 0$ one can divide by b and gets

$$y^2 = x''z' + ub^2(x'')^2 + vbx''z' + w(z')^2.$$

The derivative with respect to x'' is

$$z' + vbz' = (1+vb)z'$$

and the derivative with respect to z'

$$(1+vb)x''.$$

Now $1 + vb$ is invertible in a neighborhood of the generic point of D and thus the singularities of this conic bundle are contained in

$$x'' = z' = 0.$$

Substituting in the given equation we also get $y^2 = 0$ and thus the transformed conic bundle has smooth total space over a neighborhood of the generic point of D.

4. Discriminant profiles of conic bundles in characteristic two: an instructive example

In this Section, we work over an algebraically closed ground field k. First k may have arbitrary characteristic, later we will focus on the characteristic two case. Let X be a conic bundle over a smooth projective base B as in Definition 2.6.

Definition 4.1. We denote by $B^{(1)}$ the set of all valuations of $k(B)$ corresponding to prime divisors on B. A conic bundle $\pi\colon X \to B$ determines a Brauer class $\alpha \in \mathrm{Br}(k(B))[2]$. Moreover, we have natural maps, for $\mathrm{char}(k) \neq 2$,

$$\mathrm{Br}(k(B))[2] \xrightarrow{\oplus \partial_v} \bigoplus_{v \in B^{(1)}} H^1(k(v), \mathbb{Z}/2) \simeq \bigoplus_{v\in B^{(1)}} k(v)^\times/(k(v)^\times)^2$$

where ∂_v are the usual residue maps as in, for example, [GS06, Chaper 6], see also [Pi16, §3.1]; and for $\mathrm{char}(k) = 2$,

$$\mathrm{Br}(k(B))[2] \xrightarrow{\oplus r_v} \bigoplus_{v \in B^{(1)}} H^1(k(v), \mathbb{Z}/2) \simeq \bigoplus_{v\in B^{(1)}} k(v)/\wp(k(v))$$

where r_v is the residue map as in Definition 3.7, provided it is defined for α. In both of these case, we call the image of α in $\bigoplus_{v\in B^{(1)}} k(v)^\times/(k(v)^\times)^2$ in the first case, and in $\bigoplus_{v\in B^{(1)}} k(v)/\wp(k(v))$ in the second case, the residue profile of the conic bundle $\pi\colon X \to B$. Note that the v's for which the component in $H^1(k(v), \mathbb{Z}/2)$ of the residue profile of a conic bundle is nontrivial are a (possibly proper) subset of the divisorial valuations corresponding to the discriminant components of the conic bundle.

One main difference between characteristic not equal to 2 and equal to 2 (besides the fact that the residue profiles are governed by Kummer theory in the first case and by Artin–Schreier theory in the second case) is the following: for $\mathrm{char}(k) \neq 2$ and B, for concreteness and simplicity of exposition, a smooth projective rational surface, the residue profiles of conic bundles that can occur can be characterised as kernels of another explicit morphism, induced by further residues; more precisely, there is an exact sequence

(8)

$$0 \longrightarrow \mathrm{Br}(k(B))[2] \xrightarrow{\oplus \partial_v} \bigoplus_{v \in B^{(1)}} H^1(k(v), \mathbb{Z}/2) \xrightarrow{\oplus \partial_p} \bigoplus_{p \in B^{(2)}} \mathrm{Hom}(\mu_2, \mathbb{Z}/2)$$

where $B^{(2)}$ is the set of codimension 2 points of B, namely, the close points when S is a surface, see [A-M72, Thm. 1], [Pi16, Prop. 3.9], but also the far-reaching generalization via Bloch–Ogus–Kato complexes in [Ka86]. The maps ∂_p are also induced by residues, more precisely, if $C \subset B$ is a curve, $p \in C$ a point in the smooth locus of C, then

$$\partial_p \colon H^1(k(C), \mathbb{Z}/2) = k(C)^\times/(k(C)^\times)^2 \to \mathrm{Hom}(\mu_2, \mathbb{Z}/2)_p \simeq \mathbb{Z}/2$$

is just the valuation taking the order of zero or pole of a function in $k(C)^\times/(k(C)^\times)^2$ at p, modulo 2 (if C is not smooth at p, one has to make a slightly more refined definition involving the normalisation).

One has the fundamental result of de Jong [deJ04], [Lieb08, Thm. 4.2.2.3] (though for 2-torsion classes, it was proved earlier by Artin [Artin82, Thm. 6.2]) that for fields of transcendence degree 2 over an algebraically closed ground field k (of any characteristic), the period of a Brauer class equals the index, hence that every class in $\mathrm{Br}(k(B))[2]$ can be represented by a quaternion algebra, i.e., by a conic bundle over an open part of B.

However, in characteristic 2, we cannot expect a sequence that naïvely has similar exactness properties as the one in (8), as the following example shows.

Example 4.2. Let $X \subset \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$ be the conic bundle defined by an equation

$$Q = ax^2 + axz + by^2 + byz + cz^2 = 0,$$

where x, y, z are fibre coordinates in the "fibre copy" $\mathbb{P}^2$ in $\mathbb{P}^2 \times \mathbb{P}^2$, and a, b, c are general linear forms in the homogeneous coordinates u, v, w on the base $\mathbb{P}^2$. Then the discriminant is of degree 3 and consists of the three lines given by $a = 0$, $b = 0$ and $a = b$, which intersect in the point $P = (0 : 0 : 1)$. Indeed, if we want to find the points with coordinates $(u : v : w)$ on the base such that the fibre of the conic bundle over this point is singular, in other words, is such that there exist $(x : y : z)$ in $\mathbb{P}^2$ satisfying

$$Q_x = az = 0, \quad Q_y = bz = 0, \quad Q_z = by + ax = 0$$

and also $ax^2 + axz + by^2 + byz + cz^2 = 0$ (Euler's relation does not automatically imply the vanishing of the equation of the conic itself because the characteristic is two), then we have to look for those points $(u : v : w)$ where

$$\begin{pmatrix} a & b & 0 \\ 0 & 0 & a \\ 0 & 0 & b \\ \sqrt{a} & \sqrt{b} & \sqrt{c} \end{pmatrix}$$

has rank less than or equal to 2, which, on quick inspection, means $a = 0, b = 0$, or $a = b$.

More precisely, the conic bundle induces Artin–Schreier double covers ramified only in P on each of those lines: For $a = 0$ we have

$$Q_{a=0} = b(y^2 + yz) + cz^2$$

which describes a nontrivial Artin–Schreier cover ramified only at $b = 0$. The same happens on the line $b = 0$ and also on the line $a = b$:

$$\begin{aligned} Q_{a=b} &= a(x^2 + xz + y^2 + yz) + cz^2 \\ &= a\big((x^2 + y^2) + (x + y)z\big) + cz^2 \\ &= a\big((x + y)^2 + (x + y)z\big) + cz^2. \end{aligned}$$

The preceding example shows that we can indeed not expect a naïve analogue of the sequence (8) in characteristic 2: to define a reasonable further residue map to codimension 2 points, the only thing that springs to mind here would be to assign some measure of ramification at P for each of the three Artin–Schreier covers. But the resulting ramification measures would have to add to zero (modulo 2), and would have to be the same for each of the covers, so that only the slightly ungeometric option to assign ramification zero would remain. Note that the conic bundle in Example 4.2, when lifted to characteristic 0 by interpreting the coefficients in the defining equation in $\mathbb{Z}$, has discriminant consisting of the triangle of lines $a = 0, b = 0, 4c - a - b = 0$, with double covers over each of the lines ramified in the vertices of the triangle. That might suggest that we should define a further residue map also in characteristic 2 by using local lifts to characteristic 0 and then summing the ramification indices in those points that become identical when reducing modulo 2, an idea that is reminiscent of constructions in log geometry. But we have not succeeded in carrying this out yet.

Moreover, the theory in [Ka86], although developed also in cases where the characteristic equals the torsion order of the Brauer classes under consideration, gives no satisfactory solution either because the arithmetical Bloch–Ogus complex in [Ka86, §1] we would need to study would be the one for parameters $i = -1, q = 0$ and then condition (1.1) in loc.cit. is not satisfied, whence the further residue map we are looking for is undefined.

This seems to indicate that we have to do without a sequence such as (8), and this is exactly what we will do in Section 5: we will simply assume existence of certain Brauer classes with predefined residue profiles, and we will prove this existence in practice, such as in the examples in Section 6, by writing down conic bundles over the bases under consideration that have the sought-for residue profiles.

In fact, the next result partly explains Example 4.2 and also shows that the situation in characteristic 2 is even funnier.

Theorem 4.3. *Let $\pi\colon X \to B$ be a conic bundle in characteristic 2, where B is again a smooth projective surface. Let Δ be its discriminant. Then there is no point p of Δ locally analytically around which Δ consists of two smooth branches Δ_1, Δ_2 intersecting transversely in a point p such that above p the fibre of X is a double line, and near p, the fibres over points in $\Delta_1 \backslash \{p\}$ and $\Delta_2 \backslash \{p\}$ are two intersecting lines in $\mathbb{P}^2$.*

On the other hand, in characteristic not two, the above is the generic local normal form of the discriminant of a conic bundle around a point above which the fibre is a double line.

Proof. Let $p \in \Delta$ be a point in the discriminant. Then, as in Remark 2.7 and Definition 2.8, let $\mathbb{P}^2$ have homogeneous coordinates $(x : y : z)$ and $X_{\mathrm{univ}} \longrightarrow \mathbb{P}$ be the universal conic bundle, and let $U \subset B$ be a Zariski open neighborhood of p such that $\Delta_i \cap U \neq \emptyset$ for every irreducible component of Δ passing through p, and such that there is a morphism $f\colon U \to \mathbb{P}$ realizing $\pi|_{\pi^{-1}(U)}\colon X_{\pi^{-1}(U)} \to U$ as isomorphic to the pull-back via f of the universal conic bundle.

Besides $\Delta_{\mathrm{univ}} \subset \mathbb{P}$, there is also the locus $\mathscr{R}_1 \subset \mathbb{P}$ of double lines, defined for $\mathrm{char}(k) \neq 2$ by the vanishing of the two by two minors of the associated symmetric matrix yielding the generic conic (which coincides with the image of the Veronese embedding $\mathbb{P}^2 \to \mathbb{P}^5 = \mathbb{P}$), and for $\mathrm{char}(k) = 2$ by

$$\mathscr{R}_1 = \{a_{xy} = a_{xz} = a_{yz} = 0\}\,.$$

Let $f(p) = q$ and assume $q \in \mathscr{R}_1$; after a coordinate change we can assume (for all characteristics of k) that q has coordinates $a_{xx} = 1$ and all other coordinates equal to zero. Expanding the equation (4) locally around the point q, we get (denoting the dehomogenized affine coordinates by the same letters) the following local equation of Δ_{univ} around q (which becomes the origin in these affine coordinates)

$$a_{yz}^2 + a_{xy}^2 a_{zz} + a_{xy}a_{yz}a_{xz} + a_{xz}^2 a_{yy}. \tag{9}$$

The leading term is a_{yz}^2, whereas in characteristic not equal to 2, the same procedure applied to (3) yields

$$\left(4a_{yy}a_{zz} - a_{yz}^2\right) + a_{xy}a_{yz}a_{xz} - a_{xz}^2 a_{yy} - a_{xy}^2 a_{zz} \tag{10}$$

with leading term $4a_{yy}a_{zz} - a_{yz}^2$. Now the discriminant $\Delta \cap U$ is given, in the characteristic 2 case, by

$$f^*(a_{yz})^2 + f^*(a_{xy})^2 f^*(a_{zz}) + f^*(a_{xz})^2 f^*(a_{yy}) + f^*(a_{xy})f^*(a_{yz})f^*(a_{xz})$$

showing that the projectived tangent cone to Δ at p is either nonreduced of degree 2 or has degree at least three (in Example 4.2 the latter possibility occurs). This proves the first assertion of the Theorem.

Also notice that the local normal form ruled out in characteristic 2 by the above Theorem in a neighborhood of a point of the discriminant where the fibre is a double line, is in fact the generic local normal form in characteristic not equal to two! Indeed, by (10), the tangent cone to Δ in p is generically a cone over a nonsingular conic in $\mathbb{P}^1$, in other words, equal to two distinct lines. ☐

5. A formula for the unramified Brauer group of a conic bundle over a rational surface in characteristic two

We seek to prove the following result.

Theorem 5.1. *Let k be an algebraically closed field of characteristic 2, and let X, B be projective varieties over k, B a smooth rational surface, and let $\pi\colon X \to B$ be a conic bundle. Let $\Delta = \cup_{i\in I}\Delta_i$ be the discriminant as defined in Definition 2.8 with irreducible components Δ_i. Suppose that the geometric generic fibre of the conic bundle over each Δ_i is as in Theorem 3.11 b); in particular, assume all the Δ_i have multiplicity 1. Let $\alpha_i \in H^1(k(\Delta_i), \mathbb{Z}/2) = k(\Delta_i)/\wp(k(\Delta_i))$ be the element that the Artin–Schreier double cover induced by π on Δ_i determines.*

Assume that one can write $I = I_1 \sqcup I_2$ with both I_1, I_2 nonempty such that:

a) There exists a conic bundle $\psi\colon Y \to B$ over B, or possibly on a birational modification $B' \to B$, that induces a Brauer class in $\mathrm{Br}(k(B))$ with residue profile (in the sense of Definition 4.1) given by $(\alpha_i)_{i\in I_1} \in \bigoplus_{i\in I_1} H^1(k(\Delta_i), \mathbb{Z}/2)$, and such that for any point P in the intersection of some Δ_i and Δ_j, $i \in I_1, j \in I_2$

(in case we work on some B', this should hold for any point P lying over such an intersection), the fibre Y_P is a cross of two lines in $\mathbb{P}^2$.

b) There exist $i_0 \in I_1$ and $j_0 \in I_2$ such that α_{i_0} and α_{j_0} are nontrivial.

Then $\mathrm{Br}_{\mathrm{nr}}(k(X))[2]$ *is nontrivial.*

Note that, by the discussion following Example 4.2, the assumption a) seems hard to replace by something more cohomological or syzygy-theoretic.

Proof. Let us start the proof with a preliminary remark. By the work of Cossart and Piltant [CP08], [CP09], resolution of singularities is known for quasiprojective threefolds in arbitrary characteristic. (According to [Hi17], resolution of singularities should always hold.) Then a smooth projective model $\widetilde{X}$ of X always exists and $\mathrm{Br}_{\mathrm{nr}}(k(X))[2] = \mathrm{Br}(\widetilde{X})[2]$ holds by Theorem 3.2. Still, in all applications, for example in Section 6, we will always exhibit such a resolution explicitly.

By a result of Witt [Witt35], cf. [GS06, Thm. 5.4.1], the kernel of the natural homomorphism

$$\pi^*\colon \mathrm{Br}(k(B)) \to \mathrm{Br}(k(X))$$

is generated by the class of the conic bundle $X \to B$ itself. Denote by α that class in $\mathrm{Br}(k(B))$. Denote by β the class of $\psi\colon Y \to B$ in $\mathrm{Br}(k(B))$. We claim that $\pi^*(\beta) \in \mathrm{Br}(k(X))$ is nontrivial and unramified. It is nontrivial because $\beta \neq \alpha$ by assumption *b)*: α and β have different residues along some irreducible component Δ_{j_0} of Δ. Now to check that $\pi^*(\beta)$ is unramified, it suffices to check that for any valuation $v = v_D$ corresponding to a prime divisor D on a model $X' \to X$ which is smooth generically along D we have that $\pi^*(\beta)$ is unramified with respect to that valuation, in the sense that it is in the image of $\mathrm{Br}(\mathscr{O}_{X',D})$. Let $\Delta^{(1)} = \bigcup_{i\in I_1} \Delta_i$, $\Delta^{(2)} = \bigcup_{j \in I_2} \Delta_j$. There are two cases to distinguish:

(i) The centre Z of v on B, in other words the image of D on B, is not contained in $\Delta^{(1)} \cap \Delta^{(2)}$. In general, notice that the in general only partially defined residue map is defined for the classes β and α with respect to any divisor D' on the base B by the assumption on the geometric generic fibers of $X \to B$ over discriminant components and by Theorem 3.11. Moreover, if the centre Z is not contained in $\Delta^{(1)} \cap \Delta^{(2)}$, then β or $\beta - \alpha$ has residue zero along every divisor D' on B passing through Z. By Theorem 3.14, the class $\beta - \alpha$ comes from $\mathrm{Br}(\mathscr{O}_{B,Z})$. But $\pi^*(\beta - \alpha) = \pi^*(\beta)$, and hence $\pi^*(\beta)$ comes from $\mathrm{Br}(\mathscr{O}_{X',D})$ as desired.

(ii) The centre Z of v on B is contained in $\Delta^{(1)} \cap \Delta^{(2)}$, hence a point $Z = P$ over which the fibre Y_P is a cross of lines by the assumption in *a)* of the Theorem. Then the class $\pi^*(\beta)$ is represented by a conic bundle on X' that has a split Artin–Schreier double cover as Merkurjev residue over D by assumption *a)* of the Theorem. Hence the residue in that case is defined along D and trivial, so $\pi^*(\beta)$ comes from $\mathrm{Br}(\mathscr{O}_{X',D})$ as desired by Theorem 3.14 again.

Thus $\pi^*(\beta) \in \mathrm{Br}_{\mathrm{nr}}(k(X))[2]$ is a nontrivial class. □

6. Examples of conic bundles in characteristic two with nontrivial Brauer groups

Definition 6.1. Consider the following symmetric matrix defined over $\mathbb{Z}$

$$S = \begin{pmatrix} 2uv + 4v^2 + 2uw + 2w^2 & u^2 + uw + w^2 & uv \\ u^2 + uw + w^2 & 2u^2 + 2vw + 2w^2 & u^2 + vw + w^2 \\ uv & u^2 + vw + w^2 & 2v^2 + 2uw + 2w^2 \end{pmatrix}.$$

The bihomogeneous polynomial

$$(x, y, z)S(x, y, z)^t$$

is divisible by 2. Let $X \subset \mathbb{P}^2_{\mathbb{Z}} \times \mathbb{P}^2_{\mathbb{Z}}$ be the conic bundle defined by

$$\frac{1}{2}(x, y, z)S(x, y, z)^t = 0.$$

Here we denote by $(u : v : w)$ the coordinates of the first (base) $\mathbb{P}^2_{\mathbb{Z}}$ and by $(x : y : z)$ the coordinates of the second (fiber) $\mathbb{P}^2_{\mathbb{Z}}$.

The determinant of S is divisible by 2 so

$$D = \frac{1}{2} \det S$$

is still a polynomial over $\mathbb{Z}$. Its vanishing defines the discriminant Δ of X in the sense of Definition 2.8. We denote by $X_{(p)}$ the conic bundle over $\overline{\mathbb{F}}_p$ defined by reducing the defining equation of X modulo p. It has discriminant $\Delta_{(p)}$ defined by the reduction of D modulo p.

This example was found using the computer algebra system `Macaulay2` [M2] and Jakob Krökers Macaulay2 packages `FiniteFieldExperiments` and `BlackBoxIdeals` [Kr15].

Our aim is to prove the following result whose proof will take up the remainder of this Section.

Theorem 6.2. *The conic bundle $X \to \mathbb{P}^2$ has smooth total space that is not stably rational over $\mathbb{C}$. More precisely, X has the following properties:*

a) The discriminant $\Delta_{(p)}$ is irreducible for $p \neq 2$, hence

$$\mathrm{Br}_{\mathrm{nr}}\left(\overline{\mathbb{F}}_p\left(X_{(p)}\right)\right) = 0 \quad \text{for } p \neq 2.$$

b) The conic bundle $X_{(2)}$ satisfies the hypotheses of Theorem 5.1, hence

$$\mathrm{Br}_{\mathrm{nr}}\left(\overline{\mathbb{F}}_2\left(X_{(2)}\right)\right)[2] \neq 0.$$

c) There is a CH_0-universally trivial resolution of singularities $\sigma \colon \widetilde{X}_{(2)} \to X_{(2)}$.

Notice that the degeneration method of [CT-P16] (and [Voi15] initially, see also [To16]) shows that *b*) and *c*) imply that X is not stably rational over $\mathbb{C}$: indeed, by Theorem 3.2, we have $\mathrm{Br}(\widetilde{X}_{(2)}) = \mathrm{Br}_{\mathrm{nr}}\left(\overline{\mathbb{F}}_2\left(X_{(2)}\right)\right) \neq 0$, because of *b*); then [ABBB18, Theorem 1.1] yields that $\widetilde{X}_{(2)}$ is not CH_0-universally trivial. Finally, [CT-P16, Thm. 1.14] implies that X is not retract rational, in particular not stably rational, over $\overline{\mathbb{Q}}$, which is equivalent to saying it is not stably rational over any algebraically closed field of characteristic 0, see [KSC04, Prop. 3.33].

Moreover, item *a*) shows that the degeneration method, using reduction modulo $p \neq 2$ and the unramified Brauer group, cannot yield this result. This follows from work of Colliot-Thélène, see [Pi16, Thm 3.13, Rem. 3.14]; note that one only has to assume X is a threefold which is nonsingular in codimension 1 in [Pi16, Thm. 3.13]. Likewise, usage of differential forms as in [A-O16], see in particular their Theorem 1.1 and Corollary 1.2, does not imply the result either.

6.a. **Irreducibility of $\Delta_{(p)}$ for $p \neq 2$.** When we speak about irreducibility or reducibility in the following, we always mean geometric irreducibility or reducibility. Our first aim is to prove that $\Delta_{(p)}$ is irreducible for $p \neq 2$. This is easy for generic p since X is smooth over $\mathbb{Q}$ (by a straight-forward Gröbner basis computation [ABBBM2]). Since being singular is a codimension 1 condition, we expect that $\Delta_{(p)}$ is singular for a finite number of primes. So we need a more refined argument to prove irreducibility. Our idea is to prove that there is (counted with multiplicity) at most one singular point for each $p \neq 2$.

Lemma 6.3. *Let C be a reduced and reducible plane curve of degree at least 3 over an algebraically closed field. Then the length of the singular subscheme, defined by the Jacobi ideal on the curve, is at least 2.*

Proof. The only singularities of length 1 are those where étale locally two smooth branches of the curve cross transversely: if $f(x, y) = 0$ is a local equation for C with isolated singular point at the origin, then the length can only be 1 if

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}$$

have leading terms consisting of linearly independent linear forms. This means two smooth branches cross transversely. The only reducible curve that has only one transverse intersection is the union of two lines. □

We also need the following technical lemma.

Lemma 6.4. *Let $I \subset \mathbb{Z}[u, v, w]$ be a homogeneous ideal, $B = \{l_1, \dots, l_k\}$ a $\mathbb{Z}$-Basis of the space of linear forms $I_1 \subset I$, and M the $k \times 3$ matrix of coefficients of the l_i. Let g be the minimal generator of the ideal of 2×2 minors of M in $\mathbb{Z}$.*

If a prime p does not divide g, then I defines a finite scheme of degree at most 1 in characteristic p.

Proof. If p does not divide g there is at least one 2×2 minor m with $p \nmid m$. Therefore in characteristic p this minor is invertible and the matrix has rank at least 2. It follows that I contains at least 2 independent linear forms in characteristic p and therefore the vanishing set is either empty or contains 1 reduced point. □

Remark 6.5. Notice that the condition $p \nmid g$ is sufficient, but not necessary. For example the ideal (u^2, v^2, w^2) vanishes nowhere, but still has $g = 0$ and therefore $p | g$. The condition becomes necessary if I is saturated.

Proposition 6.6. *For $p \neq 2$, $\Delta_{(p)}$ is an irreducible sextic curve.*

Proof. We apply Lemma 6.4 to the saturation of $(D, \frac{dD}{du}, \frac{dD}{dv}, \frac{dD}{dw}) \subset \mathbb{Z}[u, v, w]$. A Macaulay2 computation gives $g = 2^{10}$ [ABBBM2]. So we have at most one singular point over $p \neq 2$ and therefore $\Delta_{(p)}$ is irreducible. □

6.b. **The unramified Brauer group of $X_{(2)}$ is nontrivial.** Let us now turn to characteristic $p = 2$.

Proposition 6.7. *We have*

$$D \equiv uw(u + w)(\gamma u + v^3) \mod 2.$$

with $\gamma = v^2 + uv + vw + w^2$. Furthermore

- *γ does not vanish at $(0 : 0 : 1)$.*
- *$\gamma u + v^3 = 0$ defines a smooth elliptic curve $E \subset \mathbb{P}^2_{\mathbb{F}_2}$.*

- *E does not contain the intersection point $(0:1:0)$ of the three lines.*
- *The intersection of E with each of the lines $w=0$ and $u+w=0$ is transverse.*
- *The line $u=0$ is an inflectional tangent to E at the point $(0:0:1)$.*

Proof. All of this is a straight forward computation. See [ABBBM2]. □

The next lemma gives us a criterion for the irreducibility of the Artin–Schreier double covers induced on the discriminant components and hence for the nontriviality of the residues of the conic bundle along these components.

Lemma 6.8. *Let $\pi\colon \overline{X} \to \mathbb{P}^2$ be a conic bundle defined over $\mathbb{F}_2$. Let $C \subset \mathbb{P}^2$ be an irreducible curve over $\mathbb{F}_2$, over which the fibers of $\overline{X}$ generically consist of two distinct lines. Let $\widetilde{C} \to C$ be the natural double cover of C induced by π. Then $\widetilde{C}$ is irreducible if the following hold:*

- *There exists an $\mathbb{F}_2$-rational point $p_1 \in C$ such that the fiber of $\overline{X}$ over p_1 splits into two lines defined over $\mathbb{F}_2$.*
- *There exists an $\mathbb{F}_2$-rational point $p_2 \in C$ such that the fiber of $\overline{X}$ over p_2 is irreducible over $\mathbb{F}_2$ but splits into two lines over $\overline{\mathbb{F}}_2$.*

Proof. Under the assumptions the double cover $\widetilde{C} \to C$ is defined over $\mathbb{F}_2$. Suppose, by contradiction, that $\widetilde{C}$ were (geometrically) reducible. Then the Frobenius morphism F would either fix each irreducible component of $\widetilde{C}$ as a set, or interchange the two irreducible components. But since C is defined over $\mathbb{F}_2$, this would mean that F either fixes each of the two lines as a set in every fiber over a $\mathbb{F}_2$-rational point of the base, or F interchanges the two lines in every fiber over a $\mathbb{F}_2$-rational point. This contradicts the existence of p_1, p_2. □

Proposition 6.9. *We consider the fibers of $X_{(2)}$ over the $\mathbb{F}_2$-rational points of the base $\mathbb{P}^2_{\overline{\mathbb{F}}_2}$ and obtain the following table:*

point	*fiber*	u	w	$u+w$	$\gamma u + v^3$
$(0:1:0)$	1 *double line*	×	×	×	
$(0:1:1)$	2 *rational lines*	×			
$(1:0:0)$	2 *rational lines*		×		×
$(1:0:1)$	2 *rational lines*			×	
$(0:0:1)$	2 *conjugate lines*	×			×
$(1:1:0)$	2 *conjugate lines*		×		
$(1:1:1)$	2 *conjugate lines*			×	

Here if a $\mathbb{F}_2$-rational point lies on a particular component of the discriminant, we put an '×' in the corresponding row and column.

Proof. All of this is again a straight forward computation. See [ABBBM2]. □

Corollary 6.10. *The conic bundle $X_{(2)}$ induces a nontrivial Artin–Schreier double cover on each component of the discriminant locus. In particular, condition b) of Theorem 5.1 is satisfied if we let I_1 index the three lines of $\Delta_{(2)}$ and I_2 the elliptic curve E.*

Proof. Use Lemma 6.8 and Proposition 6.9. □

We now want to check that the $2:1$ covers induced by $X_{(2)}$ over the three lines yield the same element in $H^1(\overline{\mathbb{F}}_2(t), \mathbb{Z}/2)$ as the $2:1$ covers in our Example 4.2. We need this to verify condition *a)* of Theorem 5.1. We only have to check that all these covers are birational to each other over the base $\mathbb{P}^1$. For this we use:

Proposition 6.11. *We work over the ground field $k = \overline{\mathbb{F}}_2$. Let $\overline{X} \subset \mathbb{P}^2 \times \mathbb{P}^2$ be a divisor of bidegree $(d, 2)$ that is a conic bundle over $\mathbb{P}^2$ via the first projection. Let furthermore $L \subset \mathbb{P}^2$ be a line in the discriminant of $\overline{X}$ such that $\overline{X}$ defines an Artin–Schreier double cover of L branched in a single reduced point R. Here the scheme-structure on R is defined by viewing it as the scheme-theoretic pull-back of the locus of double lines in the universal discriminant as in Definition 2.8. Suppose also that $\overline{X}$, L and R are defined over $\mathbb{F}_2$.*

Then either the $2:1$ cover defined over L by $\overline{X}|_L$ is trivial, or it is birational over L to the Artin–Schreier cover

$$x^2 + x + \frac{v}{u} = 0$$

where $(u : v)$ are coordinates on $L \cong \mathbb{P}^1$ such that $R = (0 : 1)$. In particular all non-trivial covers with $R = (0 : 1)$ that satisfy the above conditions yield the same element in $H^1(\overline{\mathbb{F}}_2(t), \mathbb{Z}/2)$.

Proof. Note that $\overline{X}|_L \to L$ is defined over $\mathbb{F}_2$, and defines a double cover $\pi\colon Y \to L$, where Y is the relative Grassmannian of lines in the fibres of $\overline{X}|_L \to L$. Then $Y \to L$ is also defined over $\mathbb{F}_2$ and flat over L. Hence $\mathscr{E} = \pi_*(\mathscr{O}_Y)$ is a rank 2 vector bundle on L, and Y can be naturally embedded into $\mathbb{P}(\mathscr{E})$. Then $\mathscr{E} = \mathscr{O}_L(e) \oplus \mathscr{O}_L(f)$, and Y is defined inside $\mathbb{P}(\mathscr{E})$ by an equation

$$ax^2 + bxy + cy^2 = 0$$

with a, b, c homogeneous polynomials with $\deg(a) + \deg(c) = 2\deg(b)$. Notice that $b = 0$ defines the locus of points of the base L over which the fibre is a double point. By our assumption $b = 0$ is a single reduced point. Hence $\deg(b) = 1$.

Notice that if the double cover Y is nontrivial, both a and c are nonzero, hence $\deg(a) \geq 0, \deg(c) \geq 0$ and $\deg(a) + \deg(c) = 2$.

Let $(u : v)$ be homogeneous coordinates on L. We now put $a' = a/u^{\deg(a)}, b' = b/u^{\deg(b)}, c' = c/u^{\deg c}$ and calculate over the function field of L. Apply $(x, y) \mapsto (b'x, a'y)$ to obtain

$$a'(b')^2x^2 + a'(b')^2xy + (a')^2c'y^2 = 0.$$

Divide by $a'(b')^2$ and dehomogenise via $y \mapsto 1$ to obtain the Artin–Schreier normal form

$$x^2 + x + \frac{ac}{b^2} = 0.$$

We now use the fact that we can choose the coordinates $(u : v)$ such that $b = u$. We can write $ac = \alpha u^2 + \beta uv + \gamma v^2$ with $\alpha, \beta, \gamma \in \mathbb{F}_2$:

$$x^2 + x + \alpha + \beta\frac{v}{u} + \gamma\frac{v^2}{u^2} = 0.$$

Now we use extensively the fact that we work over $\mathbb{F}_2$: firstly, either $\alpha = 0$ or $\alpha = 1$. In the second case let $\rho \in \overline{\mathbb{F}}_2$ be a root of $x^2 + x + 1$ and apply $x \mapsto x + \rho$. This gives

$$x^2 + x + \beta\frac{v}{u} + \gamma\frac{v^2}{u^2} = 0$$

in both cases. Even though the transformation was defined over $\overline{\mathbb{F}}_2$ this does not change the fact that β and γ are in $\mathbb{F}_2$.

Secondly either $\gamma = 0$ and we have

$$x^2 + x + \beta\frac{v}{u} = 0$$

or $\gamma = 1$ and we apply $x \mapsto x + \frac{v}{u}$ to obtain

$$x^2 + x + (\beta + 1)\frac{v}{u} = 0.$$

In both cases the coefficient in front of $\frac{v}{u}$ is either 0 or 1, thus the cover is either trivial or has the normal form

$$x^2 + x + \frac{v}{u} = 0.$$

□

Remark 6.12. Notice that the proof works over any field k of characteristic 2 until we have

$$x^2 + x + \beta\frac{v}{u} + \gamma\frac{v^2}{u^2} = 0.$$

Now we can eliminate γ only if it is a square in k. Even if this happens (for example if we work over $\overline{\mathbb{F}}_2$) we obtain, using $x \mapsto x + \sqrt{\gamma}(v/u)$,

$$x^2 + x + \underbrace{\left(\beta + \sqrt{\gamma}\right)}_{\beta'}\frac{v}{u} = 0.$$

So there seems to be a 1-dimensional moduli space of such covers.

Note that Propositions 6.7, 6.9, Corollary 6.10, and Proposition 6.11 together with the conic bundle exhibited in Example 4.2 show that Theorem 5.1 is applicable in the case of $X_{(2)}$, hence $\mathrm{Br}_{\mathrm{nr}}(\overline{\mathbb{F}}_2(X_{(2)}))[2] \neq 0$.

6.c. **A CH_0-universally trivial resolution of $X_{(2)}$.** Now we conclude the proof of Theorem 6.2 by showing the remaining assertion *c)*, the existence of a CH_0-universally trivial resolution of singularities $\sigma\colon \widetilde{X}_{(2)} \to X_{(2)}$.

We will use the following criterion [Pi16, Ex. 2.5 (1),(2),(3)] which summarizes results of [CT-P16] and [CTP16-2].

Proposition 6.13. *A sufficient condition for a projective morphism $f\colon V \to W$ of varieties over a field k to be CH_0-universally trivial is that the fibre V_ξ of f over every scheme-theoretic point ξ of W is a (possibly reducible) CH_0-universally trivial variety over the residue field $\kappa(\xi)$ of the point ξ. This sufficient condition in turn holds if X_ξ is a projective (reduced) geometrically connected variety, breaking up into irreducible components X_i such that each X_i is CH_0-universally trivial and geometrically irreducible, and such that each intersection $X_i \cap X_j$ is either empty or has a zero-cycle of degree 1 (of course the last condition is automatic if $\kappa(\xi)$ is algebraically closed).*

Moreover, a smooth projective retract rational variety Y over any field is universally CH_0-trivial. If Y is defined over an algebraically closed ground field, one can replace the smoothness assumption on Y by the requirement that Y be connected and each component of Y^{red} be a rational variety with isolated singular points.

We now study the behaviour of $X_{(2)}$ locally above a point P on the base $\mathbb{P}^2$, distinguishing several cases; for the cases when $X_{(2)}$ is singular locally above P, we exhibit an explicit blow-up scheme to desingularise it, with exceptional locus CH_0-universally trivial, so that Proposition 6.13 applies.

(i) The case when $P \notin \Delta_{(2)}$. In that case, $X_{(2)}$ is nonsingular locally above P.

(ii) The case when P is in the smooth locus of $\Delta_{(2)}$. In that case, $X_{(2)}$ is nonsingular locally above P as well. This can be seen by direct computation [ABBBM2].

(iii) The case when $P = (0 : 1 : 0)$ is the intersection point of the three lines $(u = 0)$, $(w = 0)$, $(u + w = 0)$ in $\Delta_{(2)}$. A direct computation shows that here $X_{(2)}$ is nonsingular locally above P as well [ABBBM2].

(iv) The case when P is one of the six intersection points of $w = 0$ or $u+w = 0$ with E. In these cases, the intersection of the two discriminant components is transverse, and the fibre above the intersection point is a cross of lines. Then $X_{(2)}$ locally above P has a CH_0-universally trivial desingularization because we have the local normal form as in Lemma 6.14 with $n = 1$, and thus, by Proposition 6.15 and Proposition 6.16, one blow-up with exceptional divisor a smooth quadric resolves the single singular point of $X_{(2)}$ above P.

(v) The case when $P = (0 : 0 : 1)$ is the point where the components $(u = 0)$ and E of $X_{(2)}$ intersect in such a way that $(u = 0)$ is an inflectional tangent to the smooth elliptic curve E. In this case, the fibre of $X_{(2)}$ above P is a cross of two conjugate lines by Proposition 6.9. We need some auxiliary results.

Lemma 6.14. *Let $\hat{\mathbb{A}}^2_{\overline{\mathbb{F}}_2}$ be the completion of $\mathbb{A}^2_{\overline{\mathbb{F}}_2}$ with affine coordinates u, v along $(0,0)$, and let $\overline{X}$ be a conic bundle over $\hat{\mathbb{A}}^2_{\overline{\mathbb{F}}_2}$. Thus $\overline{X}$ has an equation*

$$c_{xx}x^2 + c_{xy}xy + c_{yy}y^2 + c_{xz}xz + c_{yz}yz + c_{zz}z^2 = 0$$

where the c's are formal power series in u and v with coefficients in $\overline{\mathbb{F}}_2$.

Assume that

a) locally around $(0,0)$ the discriminant of $\overline{X}$ has a local equation $u(u + v^n)$, $n \geq 1$.

b) The fiber over $(0,0)$ has the form $x^2 + xy + y^2$

Then, after a change in the fibre coordinates x, y and z, we can assume the normal form

$$x^2 + xy + c_{yy}y^2 + c_{zz}z^2 = 0$$

with c_{yy} a unit, $c_{zz} = \beta u(u + v^n)$ and β a unit.

Proof. Because of assumption (*b*) we can assume that c_{xx} is a unit. After dividing by c_{xx} we can assume that we have the form

$$x^2 + c_{xy}xy + c_{yy}y^2 + c_{xz}xz + c_{yz}yz + c_{zz}z^2 = 0$$

with c_{xy} and c_{yy} units. After the substition of $x \mapsto c_{xy}x$ we can divide the whole equation by c_{xy}^2 and can assume that we have the form

$$x^2 + xy + c_{yy}y^2 + c_{xz}xz + c_{yz}yz + c_{zz}z^2 = 0$$

with c_{yy} a unit. Now substituting $x \mapsto x + c_{yz}z$ and $y \mapsto y + c_{xz}z$ we obtain the normal form

$$x^2 + xy + c_{yy}y^2 + c_{zz}z^2 = 0$$

with c_{yy} still a unit. Now the discriminant of this conic bundle ist c_{zz}. Since the discriminant was changed at most by a unit during the normalization process above, we have $c_{zz} = \beta u(u + v^n)$ as claimed. □

Proposition 6.15. *Let Y be a hypersurface in $\hat{\mathbb{A}}^4_{\overline{\mathbb{F}}_2}$ with coordinates x, y, u, v, with equation*

$$x^2 + xy + \alpha y^2 + \beta u(u + v^n) = 0, \quad n \geq 1,$$

where α and β are units in $\overline{\mathbb{F}}_2[[u, v]]$. Then Y is singular only at the origin.

Let $\widetilde{\mathbb{A}^4}$ be the blow up of $\hat{\mathbb{A}}^4_{\overline{\mathbb{F}}_2}$ in the origin and let $\widetilde{Y} \subset \widetilde{\mathbb{A}^4}$ be the strict transform of Y. If $n = 1$, then $\widetilde{Y}$ is smooth. If $n > 1$, then $\widetilde{Y}$ is singular at only one point, which we can assume to be the origin again. Around this singular point $\widetilde{Y}$ has a local equation

$$x^2 + xy + \alpha' y^2 + \beta' u(u + v^{n-1}) = 0$$

with α' and β' units in $\overline{\mathbb{F}}_2[[u, v]]$.

Proof. In $\widetilde{\mathbb{A}^4}$, we obtain 4 charts. It will turn out that in three of them $\widetilde{Y}$ is smooth and in the fourth we obtain the local equation given above.

a) $(x, y, u, v) \mapsto (x, xy, xu, xv)$ gives

$$x^2 + x^2 y + \alpha' x^2 y^2 + \beta' xu(xu + x^n v^n) = 0$$

as the total transform, and

$$1 + y + \alpha' y^2 + \beta' u(u + x^{n-1} v^n) = 0$$

as the strict transform. Notice that α' and β' are power series that only involve u, v and x. Therefore the derivative with respect to y is 1 in both cases and the strict transform is smooth in this chart.

b) $(x, y, u, v) \mapsto (xy, y, yu, yv)$ gives

$$x^2 y^2 + xy^2 + \alpha' y^2 + \beta' yu(yu + y^n v^n) = 0$$

as the total transform, and

$$x^2 + x + \alpha' + \beta' u(u + y^{n-1} v^n) = 0$$

as the strict transform. Notice that α' and β' are power series that only involve u, v and y. Therefore the derivative with respect to x is 1 in both cases and the strict transform is smooth in this chart.

c) $(x, y, u, v) \mapsto (xu, yu, u, uv)$ gives

$$x^2 u^2 + xyu^2 + \alpha' y^2 u^2 + \beta' u(u + u^n v^n) = 0$$

as the total transform, and

$$x^2 + xy + \alpha' y^2 + \beta'(1 + u^{n-1} v^n) = 0$$

as the strict transform. Notice that α' and β' are power series that only involve u, v. Therefore the derivative with respect to x and y are y and x respectively. So the singular locus lies on $x = y = 0$. Substituting this into the equation of the strict transform we get

$$\beta'(1 + u^{n-1} v^n) = 0$$

This is impossible since β' and $(1 + u^{n-1} v^n)$ are units. Therefore the strict transform is smooth in this chart.

d) $(x, y, u, v) \mapsto (xv, yv, uv, v)$ gives

$$x^2 v^2 + xyv^2 + \alpha' y^2 v^2 + \beta' uv(uv + v^n) = 0$$

as the total transform, and

$$x^2 + xy + \alpha' y^2 + \beta' u(u + v^{n-1}) = 0$$

as the strict transform. Notice that α' and β' are power series that only involve u, v. Therefore the derivative with respect to x and y are y and x respectively. So the singular locus lies on $x = y = 0$. Substituting this into the equation of the strict transform we get

$$\beta' u(u + v^{n-1}) = 0.$$

Let us now look at the derivative with respect to u:

$$\frac{d\alpha'}{du} y^2 + \frac{d\beta'}{du} u(u + v^{n-1}) + \beta' v^{n-1} = 0$$

Since $x = y = u(u + v^{n-1}) = 0$ on the singular locus, this equation reduces to $v^{n-1} = 0$. If $n = 1$, this shows that $\widetilde{Y}$ is smooth everywhere. If $n \geq 2$, we obtain that the strict transform is singular at most at $x = y = u = v = 0$ in this chart. To check that this is indeed a singular point we also calculate the derivative with respect to v:

$$\frac{d\alpha'}{dv} y^2 + \frac{d\beta'}{dv} u(u + v^{n-1}) + \beta'(n-1)uv^{n-2} = 0$$

which is automatically satisfied at $x = y = u = v = 0$.

This proves all claims of the proposition. □

Proposition 6.16. *Keeping the notation of Proposition 6.15, the exceptional divisor of $\widetilde{Y} \to Y$ is a quadric with at most one singular point.*

Proof. Recall that the equation of Y is

$$x^2 + xy + \alpha y^2 + \beta u(u + v^n) = 0.$$

We see immediately that the leading term around the origin is

$$x^2 + xy + \alpha_0 y^2 + \beta_0 u^2$$

for $n > 1$ with α_0, β_0 nonzero constants, and

$$x^2 + xy + \alpha_0 y^2 + \beta_0 u^2 + \beta_0 uv$$

for $n = 1$. The first is a quadric cone with an isolated singular point, the second is a smooth quadric. □

Summarizing, we see that Lemma 6.14, Proposition 6.15, and Proposition 6.16 show that, locally around the singular point lying above $P = (0 : 0 : 1)$, the conic bundle $X_{(2)}$ has a resolution of singularities with CH_0-universally trivial fibres. By Proposition 6.13, and taking into account cases (i)-(v) above, we conclude that $X_{(2)}$ has a CH_0-universally trivial resolution of singularities $\sigma\colon \widetilde{X}_{(2)} \to X_{(2)}$. This concludes the proof of Theorem 6.2.

References

[A-O16] H. Ahmadinezhad, T. Okada, T., *Stable rationality of higher dimensional conic bundles*, preprint (2016), arXiv:1612.04206v2 [math.AG] 19

[Artin67] E. Artin, *Algebraic Numbers and Algebraic Functions*, Gordon and Breach, New York (1967); reissued AMS Chelsea Publishing, AMS, (2006) 8, 9, 11

[Artin07] E. Artin, *Algebra with Galois Theory*, Courant Lecture Notes **15**, reprint, AMS (2007) 8

[Artin82] M. Artin, *Brauer–Severi varieties*, Brauer groups in ring theory and algebraic geometry (Wilrijk, 1981), Lecture Notes in Math., vol. 917, pp. 194–210, Springer, Berlin, 1982. 15

[A-M72] M. Artin, D. Mumford, *Some elementary examples of unirational varieties which are not rational*, Proc. London Math. Soc. (3) **25**, (1972), 75–95 15

[ACTP16] A. Auel, J.-L. Colliot-Thélène, and Parimala, *Universal unramified cohomology of cubic fourfolds containing a plane*, Brauer groups and obstruction problems: moduli spaces and arithmetic (Palo Alto, 2013), Progress in Mathematics, vol. 320, Birkhäuser Basel, 2017.

[ABBB18] A. Auel, A. Bigazzi, Chr. Böhning, H.-Chr. Graf v. Bothmer, *The universal triviality of the Chow group of* 0*-cycles and the Brauer group*, preprint, 2018. 1, 19

[ABBBM2] A. Auel, A. Bigazzi, Chr. Böhning, H.-Chr. Graf v. Bothmer, *Macaulay2 scripts for "Unramified Brauer groups of conic bundles over rational surfaces in characteristic 2"*, available at http://www.math.uni-hamburg.de/home/bothmer/m2.html. 20, 21, 24

[AB68] M. Auslander and A. Brumer, *Brauer groups of discrete valuation rings*, Nederl. Akad. Wetensch. Proc. Ser. A **71** (1968), 286–296. 12

[AG60] M. Auslander and O. Goldman, *Maximal Orders*, Transactions of the American Math. Soc. **97** (1) (1960), pp. 1–24 13

[Bogo87] F. Bogomolov, *The Brauer group of quotient spaces of linear representations*, Izv. Akad. Nauk SSSR Ser. Mat. **51**, no. 3, (1987), 485-516 5

[CL98] A. Chambert-Loir, *Cohomologie cristalline: un survol*, Exposition. Math. **16**, (1998) 333–382. 1

[Ces17] K. Cesnavicius, *Purity for the Brauer group*, preprint (2017) arXiv:1711.06456v3 [math.AG] 7

[CTO] J.-L. Colliot-Thélène, M. Ojanguren, *Variétés unirationelles non rationelles: au delà de l'exemple d'Artin et Mumford*, Invent. Math. **97** (1989), 141-158 5, 7

[CT95] J.-L. Colliot-Thélène, *Birational invariants, purity and the Gersten conjecture*, in: K-theory and Algebraic Geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992) (B. Jacob and A. Rosenberg, eds.), Proceedings of Symposia in Pure Mathematics **58.1** (1995), 1–64. 5, 6

[CT99] J.-L. Colliot-Thélène, *Cohomologie des corps valués henséliens, d'après K. Kato et S. Bloch*, in: Algebraic K-theory and its applications (Trieste, 1997), Proceedings of the Workshop and Symposium, ICTP, Trieste, Italia, 1-19 September 1997, ed. H. Bass, A. O. Kuku, C. Pedrini, World Scientific Publishing, (1999), 120–163. 10

[CTPS12] J.-L. Colliot-Thélène, R. Parimala, V. Suresh, *Patching and local-global principles for homogeneous spaces over function fields of p-adic curves*, Comment. Math. Helv. **87** (2012) 1011–1033 13

[CT-P16] J.-L. Colliot-Thélène and A. Pirutka, *Hypersurfaces quartiques de dimension 3 : non rationalité stable*, Ann. Sci. École Norm. Sup. **49** (2016), no. 2, 371–397. 19, 23

[CTP16-2] J.-L. Colliot-Thélène and A. Pirutka, *Cyclic covers that are not stably rational*, (in Russian) Izvestija RAN, Ser. Math. Tom **80** no. 4 (2016), 35–48; English translation available at here. 23

[CP08] V. Cossart and O. Piltant, *Resolution of singularities of threefolds in positive characteristic. I. Reduction to local uniformization on Artin–Schreier and purely inseparable coverings*, J. Algebra **320** (2008), no. 3, 1051–1082. 18

[CP09] V. Cossart and O. Piltant, *Resolution of singularities of threefolds in positive characteristic. II*, J. Algebra **321** (2009), no. 7, 1836–1976. 18

[deJ03] A. J. de Jong, *A result of Gabber*, preprint (2003), available at http://www.math.columbia.edu/ dejong/papers/2-gabber.pdf 5

[deJ04] A.J. de Jong, *The period-index problem for the Brauer group of an algebraic surface*, Duke Math. J. **123** (2004), no. 1, 71–94. 15

[EKM08] R., Elman, N. Karpenko, A. Merkurjev, *The Algebraic and Geometric Theory of Quadratic Forms*, AMS Colloquium Publications, vol. **56**, AMS (2008). 2, 8

[Fuji02] K. Fujiwara, *A proof of the absolute purity conjecture (after Gabber)*, in: Algebraic geometry 2000, Azumino, Adv. Stud. Pure Math. **36** (2002), pp. 153–183. 6

[Ga93] O. Gabber, *An injectivity property for étale cohomology*, Compositio Math. **86** (1993), no. 1, 1–14. 7

[Ga04] O. Gabber,*On purity for the Brauer group*, in: Arithmetic Algebraic Geometry, Oberwolfach Report **37**, (2004), 1975–1977 7

[GMS03] S. Garibaldi, A. Merkurjev, and J.-P. Serre, *Cohomological invariants in Galois cohomology*, University Lecture Series, vol. 28, AMS, Providence, R.I., 2003. 7, 8, 9, 12

[GS06] P. Gille and T. Szamuely, *Central simple algebras and Galois cohomology*, Cambridge Studies in Advanced Mathematics, vol. 101, Cambridge University Press, Cambridge, 2006. 8, 14, 18

[Gro68] A. Grothendieck, *Le groupe de Brauer I, II, III*, Dix exposés sur la cohomologie des schémas, North-Holland, Amsterdam, (1968). 5, 6

[Ha67] G. Harder, *Halbeinfache Gruppenschemata ber Dedekindringen*, Inventiones Math. **4** (1967) 165–191 13

[Hi17] H. Hironaka, *Resolution of singularities in positive characteristics*, preprint (2017), available at http://www.math.harvard.edu/ hironaka/pRes.pdf 7, 18

[ILO14] L. Illusie, Y. Laszlo, and F. Orgogozo (eds.), *Travaux de Gabber sur l'uniformisation locale et la cohomologie étale des schémas quasi-excellents*, Astérisque, no. 363–364, Société Mathématique de France, Paris, 2014. 6

[Izh91] O. Izhboldin, *On p-torsion in K_*^M for fields of characteristic p*, in: Algebraic K-theory, Adv. Sov. math. **4** (1991), 129–144 10

[Ka82] K. Kato, *Galois cohomology of complete discrete valuation fields*, Lecture Notes in Math. **67**, Springer-Verlag, Berlin (1982), 215–238 9

[Ka86] K. Kato, *A Hasse principle for two dimensional global fields*, Journal für die reine und angewandte Mathematik **366** (1986), 142–180. 9, 15, 16

[Kr15] J. Kröker, *A Macaulay2-Framework for finite field experiments for explicit and implicitly given ideals and parameter spaces*, available at `https://github.com/jakobkroeker/FiniteFieldExperiments.M2`. 19

[KSC04] J. Kollár, K.E. Smith, A. Corti, *Rational and Nearly Rational Varieties*, Cambridge Studies in Advanced Mathematics **92**, Cambridge University Press (2004) 19

[Lieb08] M. Lieblich, *Twisted sheaves and the period-index problem*, Compositio Mathematica **144** (1) (2008), 1–31 15

[M2] D. Grayson, M. Stillman, *Macaulay2, a software system for research in algebraic geometry*, available at `http://www.math.uiuc.edu/Macaulay2/` 19

[Mer15] A. S. Merkurjev, *Invariants of algebraic groups and retract rationality of classifying spaces*, preprint (2015) available at http://www.math.ucla.edu/ merkurev/publicat.htm; to appear in AMS book Algebraic Groups: Structure and Actions. 9

[Mi80] J. Milne, *Étale cohomology*, PMS **33**, Princeton University Press (1980). 7

[Pi16] A. Pirutka, *Varieties that are not stably rational, zero-cycles and unramified cohomology*, Algebraic Geometry (Salt Lake City, 2015), Proc. of Symposia in Pure Math., vol. 97, part II, pp. 459–484, 2018. 14, 15, 19, 23

[TiWa15] J-P. Tignol and A.R. Wadworth, *Value functions on simple algebras, and associated graded rings*, Springer Monographs in Math., Springer-Verlag (2015) 9

[To16] B. Totaro, *Hypersurfaces that are not stably rational*, J. Amer. Math. Soc. **29** (2016), 883–891 19

[Vac06] M. Vacquié, *Valuations and local uniformization*, in: Singularity theory and its applications, Adv. Stud. Pure Math. **43** (2006), 477–527. 6

[Voi15] C. Voisin, *Unirational threefolds with no universal codimension 2 cycle*, Invent. Math. **201** (2015), 207–237. 19

[Wei13] Ch. Weibel, *The K-book, An Introduction to Algebraic K-theory*, Graduate Studies in Math. **145**, AMS (2013) 10

[Witt35] E. Witt, *Über ein Gegenbeispiel zum Normensatz*, Math. Z. **39** (1935), no. 1, 462–467. 18

[Z-S76] O. Zariski, P. Samuel, *Commutative algebra, Volume II*, Graduate Texts in Mathematics **29**, Springer-Verlag (1976).

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