

UNIVERSAL TRIVIALITY OF THE CHOW GROUP OF 0-CYCLES AND THE BRAUER GROUP

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ABSTRACT. We prove that a smooth proper universally CH_0 -trivial variety X over a field k has universally trivial Brauer group. This fills a gap in the literature concerning the p -torsion of the Brauer group when k has characteristic p .

1. INTRODUCTION

Our main motivation for considering the Chow group of 0-cycles and the Brauer group is the rationality problem in algebraic geometry. Both groups are stable birational invariants of smooth projective varieties that have been used to great success in obstructing stable rationality for various classes of algebraic varieties. The Brauer group, and more generally, unramified cohomology groups, have been used most notably in the context of the Lüroth problem and in Noether's problem, see [A-M72], [Sa77], [Bogo87], [CTO], [CT95], [Bogo05], [Pey08]. The Chow group of 0-cycles has gained increased attention since the advent of the degeneration method due to Voisin [Voi15], and further developed by Colliot-Thélène and Pirutka [CT-P16], which unleashed a torrent of breakthrough results proving the non stable rationality of many types of conic bundles over rational bases [ABBP16], [BB16], [HKT15], [A-O16], hypersurfaces of not too large degree in projective space [To16], [Sch18], and many other geometrically interesting classes of rationally connected varieties, e.g., [CT-P16], [HT16], [HPT16]. See [Pey16] for a survey of many of these results. This method relies on the good specialisation properties of the *universal triviality* of the Chow group of 0-cycles for both equicharacteristic and mixed characteristic degenerations. The Brauer group can be, and has been, used as a tool to obstruct the universal triviality of the Chow group of 0-cycles, principally using equicharacteristic degenerations; our goal is to show that even for mixed characteristic degenerations to positive characteristic p , Brauer classes of p -primary torsion can still obstruct the universal triviality of the Chow group of 0-cycles.

Let X be a smooth proper scheme over a field k , with $\mathrm{CH}_0(X)$ its Chow group of 0-cycles and $\mathrm{Br}(X) = H_{\text{ét}}^2(X, \mathbb{G}_m)$ its (cohomological) Brauer group. Following [ACTP16] (cf. [CT-P16]), we say that X is *universally CH_0 -trivial* if the degree map $\mathrm{CH}_0(X_F) \rightarrow \mathbb{Z}$ is an isomorphism for every field extension F/k , and that the Brauer group of X is *universally trivial* if the natural map $\mathrm{Br}(F) \rightarrow \mathrm{Br}(X_F)$ is an isomorphism for every field extension F/k . A variety over a field k is an integral scheme X , separated and of finite type over k . Our main result is the following.

Theorem 1.1. *Let X be a smooth proper variety over a field k . If X is universally CH_0 -trivial, then the cohomological Brauer group of X is universally trivial.*

When k has characteristic zero (more generally, for torsion in the Brauer group prime to the characteristic), this is a result of Merkurjev [Mer08, Thm. 2.11], which

we review below. Theorem 1.1 is claimed in [CT-P16, Thm. 1.12(iii)(b)], but as it relies on the results in [Mer08], it is only proved for torsion prime to the characteristic. Our proof, which covers the case of p -primary torsion in the Brauer group when k has characteristic $p > 0$, follows a simplified version of an argument in [Mer08] utilising a pairing between the Chow group of 0-cycles and the Brauer group, but with several nontrivial new ingredients.

The result of Merkurjev [Mer08] is that for a smooth proper variety X over a field k , the universal triviality of CH_0 is equivalent to the condition that for all Rost cycle modules M (see [Ro96]) and all field extensions F/k , the subgroup $M_{\mathrm{nr}}(F(X)/F) \subset M(F(X))$ of unramified elements of the function field $F(X)$, is trivial, meaning that the natural map $M(F) \rightarrow M_{\mathrm{nr}}(F(Y)/F)$ is an isomorphism. In particular, taking M as Galois cohomology with finite torsion coefficients μ_l , where l is prime to the characteristic of k , the group of unramified elements is precisely the usual unramified cohomology groups $H_{\mathrm{nr}}^i(F(Y)/F, \mu_l^{\otimes i-1})$, see [Se97], [CTO], [CT95], [GMS03], [Bogo05], [GS06], [Pey08]. For $i = 2$, one gets the l -torsion in the (unramified) Brauer group $\mathrm{Br}_{\mathrm{nr}}(F(Y))[l]$. In particular, the presence of nontrivial unramified classes obstruct the universal triviality of CH_0 . Differential forms in positive characteristic, used with mixed characteristic degenerations, provide a different type of obstruction to the universal triviality of CH_0 , see [To16]. For new applications of Theorem 1.1 to the rationality problem, see [ABBB18].

Now suppose that k has characteristic $p > 0$, and we consider the p -torsion $\mathrm{Br}(X)[p]$ in the Brauer group of X . In fact, $\mathrm{Br}(X)[p]$ is no longer a group of unramified elements in any graded piece of any Rost cycle module, since residue maps are not generally defined, see [Mer15, Intro.], [GMS03, App. A]. Hence one can no longer appeal directly to [Mer08] to deduce the triviality of $\mathrm{Br}(X)[p]$ under the assumption that X is universally CH_0 -trivial. However, $\mathrm{Br}(X)[p]$ is isomorphic to the subgroup of “unramified classes” in $H^2(k(X), \mathbb{Z}/p(1))$, where $\mathbb{Z}/p(j)$ is defined as in [Ka86] via the logarithmic part of the de Rham–Witt complex. More generally, we have the following.

Problem 1.2. *Let X be a smooth proper variety over an algebraically closed field k of characteristic $p > 0$. Assume that X is universally CH_0 -trivial. Is the subgroup of “unramified classes” in the Galois cohomology group $H^i(k(X), \mathbb{Z}/p(j))$, i.e., those coming from $H_{\mathrm{ét}}^i(\mathrm{Spec}(A), \mathbb{Z}/p(j))$ for every discrete valuation ring A with fraction field $k(X)$, trivial?*

Our main result gives a positive solution to Problem 1.2 when $i = 2$ and $j = 1$, namely, for the p -torsion in the Brauer group. Note that Problem 1.2 is of a similar type as many problems that crystalline cohomology was developed for, see [CL98].

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2. BACKGROUND ON BRAUER GROUPS

All schemes below will be assumed to be Noetherian and k will denote a field unless otherwise specified. For a scheme X , we will denote by $\mathrm{Br}(X) = H_{\mathrm{ét}}^2(X, \mathbb{G}_m)$ the cohomological Brauer group of X , the étale cohomology group of X with values in the sheaf of units $\mathbb{G}_{m,X}$. For a commutative ring A , we also write $\mathrm{Br}(A)$ for the cohomological Brauer groups of $\mathrm{Spec} A$.

The usual Brauer group of Azumaya algebras on X , considered up to Morita equivalence, is a subgroup of $H_{\text{ét}}^2(X, \mathbb{G}_m)$. If X is quasi-projective over a base ring (more generally, admits an ample invertible sheaf), then it is a result of Gabber [deJ03] that these two groups coincide. The analogous result over a smooth proper variety over a field is still unknown.

If X is integral with function field K , then restriction to the generic point induces a map $\text{Br}(X) \rightarrow \text{Br}(K)$, which is an injection when X is regular (more generally, geometrically locally factorial), see [Gro68, II, Prop. 1.4].

We now summarize some functorial properties of the Brauer group that we need.

Proposition 2.1. *The cohomological Brauer group has the following properties.*

a) *For any morphism $g : X \rightarrow Y$, there is an induced pull-back map*

$$g^* : \text{Br}(Y) \rightarrow \text{Br}(X).$$

b) *For a finite flat morphism $f : Y' \rightarrow Y$, there is a push-forward or corestriction map*

$$\text{cor}_{Y'/Y} : \text{Br}(Y') \rightarrow \text{Br}(Y),$$

also denoted by f_ or $\text{Nm}_{Y'/Y}$, which satisfies the following properties:*

i) *For a composition of finite flat morphisms $Y'' \rightarrow Y' \rightarrow Y$, we have*

$$\text{cor}_{Y''/Y} = \text{cor}_{Y'/Y} \circ \text{cor}_{Y''/Y'}.$$

ii) *For a finite flat morphism $f : Y' \rightarrow Y$ of degree d and $\alpha \in \text{Br}(Y)$, we have*

$$f_* f^* \alpha = d\alpha.$$

iii) *Given a finite flat morphism $f : Y' \rightarrow Y$, a morphism $g : X \rightarrow Y$, and a fiber square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & Y' \\ \downarrow f' & & \downarrow f \\ X & \xrightarrow{g} & Y. \end{array}$$

then for any $\beta \in \text{Br}(Y')$, we have

$$g^* f_* \beta = (f')_* (g')^* \beta.$$

Proof. Part a) is obvious by the contravariance of étale cohomology. Part b) is due to Deligne [SGA4, Exp. XVII 6.3.13], see also [ISZ11, Thm. 1.4], [Preu13, p. 453]. For any finite flat morphism $f : Y' \rightarrow Y$, there exists a natural homomorphism $f_* \mathbb{G}_{m, Y'} \rightarrow \mathbb{G}_{m, Y}$ of sheaves of abelian groups on Y , cf. [Mum66, Lecture 10]. Taking étale cohomology, we arrive at a homomorphism

$$\gamma : H_{\text{ét}}^2(Y', f_* \mathbb{G}_{m, Y'}) \rightarrow H_{\text{ét}}^2(Y, \mathbb{G}_m).$$

The Leray spectral sequence for the map $f : Y' \rightarrow Y$ yields a map

$$\iota : H_{\text{ét}}^2(Y, f_* \mathbb{G}_{m, Y'}) \rightarrow H_{\text{ét}}^2(Y', \mathbb{G}_m),$$

which is an isomorphism because $R^i f_* \mathbb{G}_{m, Y'} = 0$ for all $i > 0$ as f is finite. Putting $\text{cor}_{Y'/Y} := \gamma \circ \iota^{-1}$, we obtain a corestriction map on Brauer groups. This has property i) by construction, ii) since the composition $\mathbb{G}_{m, Y} \rightarrow f_* \mathbb{G}_{m, Y'} \rightarrow \mathbb{G}_{m, Y}$ with the unit of adjunction is multiplication by the degree of f , and iii) by the compatibility of the norm map with base change, of the Leray spectral sequence with pull-backs, and of the Leray spectral sequence with change of coefficients, see also [Preu13, §4]. \square

Definition 2.2. If $i : x \hookrightarrow X$ is the inclusion of a point with residue field $k(x)$ into X , then for an $\alpha \in \text{Br}(X)$ we denote $i^*(\alpha) \in \text{Br}(k(x))$ simply by $\alpha(x)$ and call it the value of α at x .

Definition 2.3. Let $\varphi : X \rightarrow Y$ be a finite surjective morphism of k -varieties, and let $f \in k(X)$ and $g \in k(Y)$ be rational functions. We denote by $\varphi^*(g)$ the pull-back of the rational function g to X , and by $\text{Nm}_{X/Y}(f)$, or $\varphi_*(f)$, the norm of the rational function f to Y . The later is defined as follows: via φ^* , there is a finite extension of fields $k(X)/k(Y)$, and $\text{Nm}_{X/Y}(f)$ is the determinant of the endomorphism of the $k(Y)$ -vector space $k(X)$ that is given by multiplication by f .

In applications, when one wants to determine $\text{Br}(X)$ for some smooth projective k -variety X , one is frequently only given some singular model of X a priori: it thus becomes desirable, especially since some models of such X can be highly singular and difficult to desingularise explicitly, to determine $\text{Br}(X)$ in terms of data associated to the function field $k(X)$ only. This is really the idea behind unramified invariants as for example in [Bogo87], [CTO], [CT95]. We have an inclusion

$$\text{Br}(X) \subset \text{Br}(k(X))$$

by [Gro68, II, Cor. 1.10], given by pulling back to the generic point of X . One wants to single out the classes inside $\text{Br}(k(X))$ that belong to $\text{Br}(X)$ in valuation-theoretic terms. Since the basic reference [CT95] for this often works under the assumption that the torsion orders of the classes in the Brauer group be coprime to the characteristic of k , we state and prove below a result in the generality we need here, although most of its ingredients are scattered in the available literature.

Basic references for valuation theory are [Z-S76] and [Vac06]. Valuation here without modifiers such as “discrete rank 1” etc. means a general Krull valuation.

Definition 2.4. Let X be a smooth proper variety over a field k and let \mathbf{S} be a subset of the set of all Krull valuations of the function field $k(X)$ of X . All the valuations we will consider below will be geometric in the sense that they are assumed to be trivial on the ground field k . For $v \in \mathbf{S}$, we denote by $A_v \subset k(X)$ the valuation ring of v . Then we denote by $\text{Br}_{\mathbf{S}}(k(X)) \subset \text{Br}(k(X))$ the set of all those Brauer classes $\alpha \in \text{Br}(k(X))$ that are in the image of the natural map $\text{Br}(A_v) \rightarrow \text{Br}(k(X))$ for all $v \in \mathbf{S}$. Specifically, we will consider the following examples of sets \mathbf{S} .

- a) The set **DISC** of discrete rank 1 valuations of $k(X)$ with fraction field $k(X)$.
- b) The set **DIV** of all divisorial valuations of $k(X)$ corresponding to some prime divisor D on a model X' of $k(X)$, where X' is assumed to be generically smooth along D .
- c) The set **DIV/ X** of all divisorial valuations of $k(X)$ corresponding to a prime divisor on X .

We denote the corresponding subgroups of $\text{Br}(k(X))$ by

$$\text{Br}_{\text{DISC}}(k(X)), \text{Br}_{\text{DIV}}(k(X)), \text{Br}_{\text{DIV}/X}(k(X))$$

accordingly. In addition, we define

$$\text{Br}_{\text{LOC}}(k(X))$$

as those classes in $\text{Br}(k(X))$ coming from $\text{Br}(\mathcal{O}_{X,x})$ for every (scheme-)point $x \in X$.

Note the containments $\text{DISC} \supset \text{DIV} \supset \text{DIV}/X$, which are all strict in general: for the first, recall that divisorial valuations are those discrete rank 1 valuations v with the property that the transcendence degree of their residue field is $\dim X - 1$ [Z-S76,

Ch. VI, §14, p. 88], [Vac06, §1.4, Ex. 5]; and that there are discrete rank 1 valuations that are not divisorial, for example, so-called analytic arcs [Vac06, Ex. 8(ii)].

Theorem 2.5. *Let X be a smooth proper variety over a field k . Then all of the natural inclusions*

$$\text{Br}(X) \subset \text{Br}_{\text{DISC}}(k(X)) \subset \text{Br}_{\text{DIV}}(k(X)) \subset \text{Br}_{\text{DIV}/X}(k(X))$$

are equalities.

To prove this, we need two preliminary results. The first is a purity result for the cohomological Brauer group of a variety over a field.

Theorem 2.6. *Let V be a smooth k -variety, and let $U \subset V$ be an open subvariety such that $V - U$ has codimension ≥ 2 in V . Then the restriction $\text{Br}(V) \rightarrow \text{Br}(U)$ is an isomorphism.*

Proof. The proof starts with a reduction to the case of the punctured spectrum of a strictly Henselian regular local ring of dimension ≥ 2 . For torsion prime to the characteristic of k , this follows from the absolute cohomological purity conjecture, whose proof, due to Gabber, appears in [Fuji02] or [ILO14]. For the p primary torsion when k has characteristic p , this follows from [Ga93, Thm. 2.5]. See also [Ga04, Thm. 5] and its proof. Recently, purity for the cohomological Brauer group has been established in complete generality over any scheme [Ces17]. \square

The second is a standard Meyer–Vietoris exact sequence for étale cohomology.

Theorem 2.7. *Let V be a scheme. Suppose that $V = U_1 \cup U_2$ is the union of two open subsets. For any sheaf \mathcal{F} of abelian groups in the étale topology on V there is a long exact cohomology sequence*

$$\begin{aligned} 0 \rightarrow H_{\text{ét}}^0(V, \mathcal{F}) &\rightarrow H_{\text{ét}}^0(U_1, \mathcal{F}) \oplus H_{\text{ét}}^0(U_2, \mathcal{F}) \rightarrow H_{\text{ét}}^0(U_1 \cap U_2, \mathcal{F}) \\ &\rightarrow H_{\text{ét}}^1(V, \mathcal{F}) \rightarrow H_{\text{ét}}^1(U_1, \mathcal{F}) \oplus H_{\text{ét}}^1(U_2, \mathcal{F}) \rightarrow H_{\text{ét}}^1(U_1 \cap U_2, \mathcal{F}) \rightarrow \dots \end{aligned}$$

which is functorial in \mathcal{F} .

Proof. See [Mi80, Ex. 2.24, p. 110]. \square

Proof (of Theorem 2.5). Note that to ensure that one has the inclusion $\text{Br}(X) \subset \text{Br}_{\text{DISC}}(k(X))$ one uses the valuative criterion for properness so that every valuation on $k(X)$ is centered at a point of X . The inclusion $\text{Br}(X) \subset \text{Br}_{\text{DIV}/X}(k(X))$ holds regardless of properness assumptions, and the inclusions $\text{Br}_{\text{DISC}}(k(X)) \subset \text{Br}_{\text{DIV}}(k(X)) \subset \text{Br}_{\text{DIV}/X}(k(X))$ come immediately from the definitions.

We prove that every class α in $\text{Br}_{\text{DIV}/X}(k(X))$ belongs to $\text{Br}(X)$. Any class α in $\text{Br}(k(X))$ can be represented by a class, denoted α_V , in $\text{Br}(V)$ where $V \subset X$ is open with complement a union of prime divisors D_i on X . Moreover, as α is in the image of $\text{Br}(\mathcal{O}_{X, \xi_i})$ for ξ_i the generic point of D_i and all i , we have that there are open subsets U_i of X such that $U_i \cap D_i \neq \emptyset$ and there exist classes $\alpha_{U_i} \in \text{Br}(U_i)$ whose images in $\text{Br}(k(X))$ agree with α . By Theorem 2.7, using the cohomological description of the Brauer group, we get that there exist an open subset $U \subset X$ with complement $X \setminus U$ of codimension at least 2 and a class $\alpha_U \in \text{Br}(U)$ inducing α in $\text{Br}(k(X))$. Note that this uses Grothendieck’s injectivity result that Brauer classes on a regular integral scheme agreeing at the generic point agree everywhere to achieve the gluing: to use Theorem 2.7 repeatedly, we have to ensure that at each step the Brauer classes given on the two open sets $\Omega_1 = V \cup U_1 \cup \dots \cup U_{i-1}$ and

$\Omega_2 = U_i$ agree on all of $\Omega_1 \cap \Omega_2$ and this follows from Grothendieck's result since we know they agree in $\text{Br}(k(X))$.

Finally, once we have extended α to the open subset U with $\text{codim}(X \setminus U, X) \geq 2$, Theorem 2.6 shows that α comes from $\text{Br}(X)$. \square

Remark 2.8. In the setting of Theorem 2.5, we will agree to denote the group $\text{Br}_{\text{DIV}}(k(X))$ by $\text{Br}_{\text{nr}}(k(X))$ and call this the unramified Brauer group of the function field $k(X)$. We will also use this notation for potentially singular X in positive characteristic. According to [Hi17], a resolution of singularities should always exist, but we do not need this result: in all our applications we produce explicit resolutions \tilde{X} , and then we know $\text{Br}_{\text{nr}}(k(X)) = \text{Br}(\tilde{X})$.

3. A VARIANT OF WEIL RECIPROCITY AND A PAIRING

Let V be a proper variety over a field k (not necessarily algebraically closed in the sequel). Let $Z_0(V)$ be the group of 0-cycles on V and $\text{CH}_0(V)$ the Chow group of 0-cycles on V up to rational equivalence. We first define a pairing

$$Z_0(V) \times \text{Br}(V) \rightarrow \text{Br}(k)$$

as follows: for a 0-cycle $z = \sum_i a_i z_i$ and a Brauer class $\alpha \in \text{Br}(V)$, define

$$\langle z, \alpha \rangle = \alpha(z) = \sum_i a_i \text{cor}_{k(z_i)/k}(\alpha(z_i))$$

where a_i are integers, z_i are closed points of V with residue fields $k(z_i)$, and cor is the corestriction map. Thus $\langle z, \alpha \rangle$ and $\alpha(z)$ are simply alternative notation for this pairing which are convenient in different settings. This pairing is clearly bilinear. The main result of this section is the following.

Proposition 3.1. *Let X be a smooth proper variety over a field k . The pairing on 0-cycles defined above factors through to a pairing*

$$\text{CH}_0(X) \times \text{Br}(X) \rightarrow \text{Br}(k).$$

We need to show that rationally equivalent 0-cycles on X have the same values when paired with Brauer classes. The 0-cycles on X rationally equivalent to zero are sums of cycle of the form $\pi_*(z)$, where $D \subset X$ is a curve with normalization

$$\pi : C \rightarrow D$$

and $z = \text{div}(f)$ for a rational function $f \in k(C)^*$.

Lemma 3.2. *If $\pi : C \rightarrow D$ is any finite morphism of proper curves, $\alpha \in \text{Br}(D)$, $z \in Z_0(C)$, then*

$$\langle z, \pi^*(\alpha) \rangle = \langle \pi_*(z), \alpha \rangle.$$

Proof. Since the pairing is bilinear and π_* linear, it suffices to prove this for the case where $z \in C$ is a single closed point with residue field $k(z)$. The image $w = \pi(z)$ (of the map π on the underlying point sets of C and D) is a closed point of D and we have a commutative diagram

$$\begin{array}{ccccc} \text{Spec } k(z) & \xrightarrow{\iota_z} & C & & \\ \epsilon_z \swarrow & & \downarrow \pi & & \downarrow \pi \\ \text{Spec } k & \xleftarrow{\epsilon_w} & \text{Spec } k(w) & \xrightarrow{\iota_w} & D \end{array}$$

The left hand side $\langle z, \pi^*(\alpha) \rangle$ is nothing but $\epsilon_{z,*}(\iota_z^* \pi^* \alpha)$, which we can rewrite as

$$\epsilon_{z,*} \iota_z^* \pi^* \alpha = \epsilon_{w,*} \pi_{z,*} \pi_z^* \iota_w^* \alpha = [k(z) : k(w)] \epsilon_{w,*} \iota_w^* \alpha = \langle \pi_*(z), \alpha \rangle$$

noting that $\epsilon_{z,*} = \epsilon_{w,*} \pi_{z,*}$ and that $\pi_{z,*} \pi_z^*$ is multiplication by $[k(z) : k(w)]$ by Proposition 2.1b, and since $\pi_*(z) = [k(z) : k(w)]w$. \square

We point out that the previous Lemma 3.2 holds more generally for any finite morphism of smooth proper varieties. Now, the proof of Proposition 3.1 will be complete once we establish the following.

Proposition 3.3. *Let C be a normal proper curve over k . Then for any rational function $f \in k(C)^*$ and any Brauer class $\alpha \in \text{Br}(C)$, we have that $\alpha(\text{div}(f)) = 0$.*

Note that if α has order l coprime to the characteristic of k , then one can write

$$\alpha(\text{div}(f)) = \sum_{P \in C^{(1)}} \text{cor}_{k(P)/k}(\text{ord}_P(f) \alpha(P)) = \sum_{P \in C^{(1)}} \text{cor}_{k(P)/k} \partial_P((f) \cup \alpha)$$

where the sum is over all codimension 1 points of C , where ord_P is the order of vanishing, and where ∂_P is the residue map associated to the discrete valuation on $k(C)$ determined by P . Hence, Proposition 3.3 follows from the Weil reciprocity formula in Rost [Ro96, Prop. 2.2 (RC)]. See also [GS06, Proposition 7.4.4] for a general Weil reciprocity law in Milnor K -theory. For torsion a power of $\text{char}(k)$, the theory of residue maps breaks down and only some remnants remain [GMS03, App. A, p. 152ff]; the proof in [Ro96, p. 340–341] heavily relies on residues as well as co-residues that we do not know how to define in this situation, so we resort to a more elementary approach to prove the special case of Weil reciprocity needed in Proposition 3.3. In fact, the proof is modeled on the proof of classical Weil reciprocity on a smooth projective curve C : for nonzero rational functions f, g with disjoint zero and polar sets $f(\text{div}(g)) = g(\text{div}(f))$, see [ACGH85, p. 283].

Proof (of Proposition 3.3). The pairing above induces a bilinear pairing

$$k(C)^* \times \text{Br}(C) \rightarrow \text{Br}(k), \quad (f, \alpha) \mapsto \langle f, \alpha \rangle$$

by writing $\text{div}(f) = \sum_i a_i z_i$ and defining $\langle f, \alpha \rangle = \sum_i a_i \text{cor}_{k(z_i)/k}(\alpha(z_i))$ as above. We want to show that this pairing is trivial on $k(C)^* \times \text{Br}(C)$.

Step 1. We prove Proposition 3.3 for $C = \mathbb{P}_k^1$. In that case, α is induced from $\text{Br}(k)$; more precisely, if $\omega : \mathbb{P}_k^1 \rightarrow \text{Spec}(k)$ is the structure morphism, $\alpha = \omega^*(\alpha')$ from some $\alpha' \in \text{Br}(k)$: indeed, a theorem due to Grothendieck says that $\text{Br}(\mathbb{P}_k^1) = \text{Br}(k)$ for any field; alternatively, by [Mer15, Prop. 3.4 (4)], we have that the natural morphism $H^2(k, \mathbb{Q}/\mathbb{Z}(1)) \rightarrow H_{\text{nr}}^2(k(\mathbb{P}_k^1), \mathbb{Q}/\mathbb{Z}(1))$ is an isomorphism, and by [Mer15, §2.1], $H^2(F, \mathbb{Q}/\mathbb{Z}(1)) = \text{Br}(F)$ for any field. Moreover, $\text{Br}(X)$ injects into $\text{Br}_{\text{nr}}(k(X))$ for any regular integral k -variety [Gro68, II, Cor. 1.10], with the definition of unramified elements as in [Mer15, §2.4].

Now if z_i is the image of $\psi_i : \text{Spec}(k(z_i)) \rightarrow X$, then the pull-back of α' via the composition

$$\text{Spec}(k(z_i)) \xrightarrow{\psi_i} X \xrightarrow{\omega} \text{Spec}(k)$$

followed by the corestriction $\text{cor}_{k(z_i)/k} : \text{Br}(k(z_i)) \rightarrow \text{Br}(k)$ is nothing but $[k(z_i) : k] \alpha'$. Hence,

$$\langle f, \alpha \rangle = \sum_i a_i \text{cor}_{k(z_i)/k}(\alpha(z_i)) = \alpha' \sum_i [k(z_i) : k] a_i = \text{deg}(\text{div}(f)) \alpha' = 0$$

since the degree of a principal divisor is zero.

Step 2. Reduction of the general case to \mathbb{P}_k^1 . Given a nonconstant $f \in k(C)^*$ (for constant f 's the assertion to be proved is trivial), it defines a surjective finite flat morphism $C \rightarrow \mathbb{P}_k^1$, which we will denote by φ_f . Moreover, letting $g = \text{id}_{\mathbb{P}_k^1} \in k(\mathbb{P}_k^1)$ we have the very stupid but useful equality $\varphi_f^*(g) = f$. Now by Lemma 3.4, proven below, to conclude, we can simply compute

$$\langle f, \alpha \rangle = \langle \varphi_f^*(g), \alpha \rangle = \langle g, (\varphi_f)_*(\alpha) \rangle = 0,$$

the last equality by Step 1. \square

Lemma 3.4. *For a finite flat covering of normal proper curves $\varphi : C \rightarrow D$ over k , a 0-cycle $z \in Z_0(D)$, and a Brauer class $\alpha \in \text{Br}(C)$, we have*

$$\langle \varphi^*(z), \alpha \rangle = \langle z, \varphi_*(\alpha) \rangle$$

where $\varphi_*(\alpha) = \text{Nm}_{C/D}(\alpha)$ is the norm map of Proposition 2.1 and $\varphi^*(z)$ the pull-back of divisors from Definition 2.3.

Proof. We divide the proof into steps for greater transparency. By the bilinearity of the pairing and linearity of φ^* , we can reduce to the case when z is a closed point in D with residue field $k(z)$, which we will assume in the sequel.

Step 1. Applying Proposition 2.1 b iii, to

$$\begin{array}{ccc} C_z & \longrightarrow & C \\ \downarrow \varphi_z & & \downarrow \varphi \\ \text{Spec } k(z) & \longrightarrow & D \end{array}$$

where C_z is the scheme-theoretic fiber over z , we see that it suffices to prove the equality in Lemma 3.4 for φ replaced by φ_z , α replaced by α_z , the restriction of α to C_z . In the following we drop the subscripts z again for ease of notation.

Step 2. Abbreviating $k(z)$ to L , the fiber C_z in Step 1 is a disjoint union of schemes of the form $Y_x := \text{Spec } \mathcal{O}_{C,x}/((\pi_x)^{n_x})$ where $x \in C$ is a closed point with local ring the discrete valuation ring $\mathcal{O}_{C,x}$, with uniformiser π_x , $n_x \in \mathbb{N}$, and residue field $k(x)$ some finite extension field of L . Here x runs over the closed points lying above z . By the bilinearity of the pairing and linearity of φ_* again, it thus suffices to prove: given

$$(1) \quad \begin{array}{ccc} & Y_x & \xleftarrow{\text{red}} \text{Spec } \mathcal{O}_{C,x}/(\pi_x) = \text{Spec}(k(x)) \\ & \swarrow \varphi & \searrow \psi \\ \text{Spec } k & \longleftarrow \text{Spec } L & \end{array}$$

and a Brauer class α in $\text{Br}(Y_x)$, then $\varphi_*(\alpha)$ is the same as n times $\psi_*(\text{red}^*(\alpha))$. Here red is the reduction map, and ψ is induced by the inclusion of fields $L \subset k(x)$. We will also sometimes find it convenient to write $(Y_x)_{\text{red}}$ for $\text{Spec}(k(x))$ in the argument below.

Step 3. We prove the remaining assertion of Step 2. To ease notation, we drop the subscript x , and denote $\mathcal{O}_{C,x}$ simply by \mathcal{O} . Let α be a Brauer class in $\text{Br}(\mathcal{O}/(\pi^n))$, for the given discrete valuation ring \mathcal{O} (which contains the field L) with residue field $\kappa = \mathcal{O}/(\pi)$ being a finite extension of L , $[\kappa : L] = d$. Let $\bar{b}_1, \dots, \bar{b}_d$ be an L -basis of κ , and let b_1, \dots, b_d be lifts of these basis elements to \mathcal{O} . Then the elements

$$b_1, \dots, b_d, \pi b_1, \dots, \pi b_d, \dots, \pi^{n-1} b_1, \dots, \pi^{n-1} b_d$$

are an (ordered) L -basis of $\mathcal{O}/(\pi^n)$. Abbreviate $\mathcal{A} = \mathcal{O}/(\pi^n)$ and write an element $a \in \mathcal{A}$ in that given basis:

$$\begin{aligned} a &= a_{0,1}b_1 + \cdots + a_{0,d}b_d \\ &\quad + (a_{1,1}b_1 + \cdots + a_{1,d}b_d)\pi \\ &\quad \vdots \\ &\quad + (a_{n-1,1}b_1 + \cdots + a_{n-1,d}b_d)\pi^{n-1}. \end{aligned}$$

Let $\kappa = \bar{\mathcal{A}}$ be the reduction of \mathcal{A} , and let $\bar{a} \in \bar{\mathcal{A}}$ be the image of a , which can consequently be written as

$$\bar{a} = a_{0,1}\bar{b}_1 + \cdots + a_{0,d}\bar{b}_d$$

in the (ordered) L -basis $\bar{b}_1, \dots, \bar{b}_d$ of $\bar{\mathcal{A}}$.

Denote by m_a and $m_{\bar{a}}$ the L -linear maps in \mathcal{A} and $\bar{\mathcal{A}}$ given by multiplication by a and \bar{a} . If in the chosen ordered basis above, $m_{\bar{a}}$ has a matrix M , then in the chosen ordered basis of \mathcal{A} the map m_a will be represented by a block lower triangular matrix of the form

$$\begin{pmatrix} M & 0 & 0 & \cdots & 0 \\ N_{21} & M & 0 & \cdots & 0 \\ N_{31} & N_{32} & M & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ N_{n1} & N_{n2} & \cdots & N_{n,n-1} & M \end{pmatrix}$$

where M and all matrices N_{ij} are $d \times d$ matrices with entries in L . In particular,

$$\text{Nm}_{\mathcal{A}/L}(a) = \left(\text{Nm}_{\bar{\mathcal{A}}/L}(\bar{a}) \right)^n.$$

We can rephrase this by saying that in the set-up of the diagram (1), we have that the norm map

$$(2) \quad N : \varphi_* \mathbb{G}_{m, Y_x} \rightarrow \mathbb{G}_{m, \text{Spec}(L)}$$

can be factored into three maps:

$$(3) \quad \varphi_* \mathbb{G}_{m, Y_x} \xrightarrow{\varrho} \varphi_* (\text{red}_* \mathbb{G}_{m, (Y_x)_{\text{red}}}) \xlongequal{\quad} \psi_* (\mathbb{G}_{m, (Y_x)_{\text{red}}}) \xrightarrow{\bar{a} \rightarrow \bar{a}^n} \psi_* (\mathbb{G}_{m, (Y_x)_{\text{red}}}) \downarrow \bar{N} \mathbb{G}_{m, \text{Spec}(L)}$$

where ϱ is induced by the reduction map red and \bar{N} is again a norm map. Applying $H_{\text{ét}}^2(\text{Spec}(L), -)$ to this sequence of homomorphism of sheaves of (multiplicative) abelian groups in the étale topology, and noting that

$$H_{\text{ét}}^2(\text{Spec}(L), \varphi_* (\mathbb{G}_{m, Y_x})) \simeq \text{Br}(Y_x)$$

and

$$H_{\text{ét}}^2(\text{Spec}(L), \psi_* (\mathbb{G}_{m, (Y_x)_{\text{red}}})) \simeq \text{Br}((Y_x)_{\text{red}})$$

by the Leray spectral sequence and the finiteness of φ and ψ , as in the proof of Proposition 2.1, we obtain that for a Brauer class α in $\text{Br}(Y_x)$, the norm $\varphi_*(\alpha)$ is indeed the same as n times $\psi_*(\text{red}^*(\alpha))$. \square

4. PROOF OF THEOREM 1.1

Let X be a smooth proper variety over a field k . Denote by $\omega : X \rightarrow \text{Spec}(k)$ the structure morphism. For any field extension F/k , we have the above pairing

$$\text{CH}_0(X_F) \times \text{Br}(X_F) \rightarrow \text{Br}(F)$$

and we will mostly be interested in the case when $F = k(X)$.

Let η be the generic point of X , considered as a closed point in $X_{k(X)}$. Then “pairing with η ” defines a map $\text{Br}(X) \rightarrow \text{Br}(k(X))$ by pulling $\alpha \in \text{Br}(X)$ back to $\text{Br}(X_{k(X)})$ and then pairing with η , thought of as an element in $\text{CH}_0(X_{k(X)})$. This map coincides with the usual map restricting a Brauer class to its own function field, which is injective for X smooth by [Gro68, II, Cor. 1.10].

Now we assume that X is universally CH_0 -trivial and proceed to prove Theorem 1.1, namely that $\text{Br}(k) = \text{Br}(X)$. The same proof shows that $\text{Br}(F) = \text{Br}(X_F)$ for any extension field F/k , i.e., that the Brauer group of X is universally trivial. By assumption, X has a 0-cycle z_0 of degree 1.

We can assume that the support of z_0 consists of closed points whose residue fields are separable extensions of k . Indeed, answering a question of Lang and Tate from the late 1960s, it is a result of Gabber, Liu, and Lorenzini [GLL13, Theorem 9.2] that a regular generically smooth nonempty scheme of finite type over k (e.g., our smooth proper variety X over k) admits a 0-cycle of minimal positive degree supported on closed points with separable residue fields. In our case, the minimal positive degree of a 0-cycle on X is 1.

Let $\alpha \in \text{Br}(X)$ and define $\alpha_0 = \alpha(z_0) \in \text{Br}(k)$. Then $(\alpha - \omega^*(\alpha_0))(z_0) = 0$ because $(\omega^*(\alpha_0))(z_0) = \deg(z_0) \cdot \alpha_0 = \alpha_0$.

Let $z'_0 \in \text{CH}_0(X_{k(X)})$ be the 0-cycle that z_0 determines on $X_{k(X)}$. Since X is universally CH_0 -trivial, we have that $\eta = z'_0$ in $\text{CH}_0(X_{k(X)})$, as they are both 0-cycles of degree 1. Denote by α' and α'_0 the pull-backs of α and $\omega^*(\alpha_0)$ to $\text{Br}(X_{k(X)})$. Then

$$0 = (\alpha' - \alpha'_0)(\eta - z'_0) = (\alpha' - \alpha'_0)(\eta) - (\alpha' - \alpha'_0)(z'_0) = (\alpha' - \alpha'_0)(\eta)$$

by bilinearity and since $(\alpha' - \alpha'_0)(z'_0)$ is the pull-back from $\text{Br}(k)$ to $\text{Br}(k(X))$ of $(\alpha - \omega^*(\alpha_0))(z_0)$, which we know is zero. Here we needed the fact that z_0 is supported on closed points whose residue fields are separable extensions of k so that z_0 pulled back to $X_{k(X)}$ is supported on reduced closed points, and hence restricting a Brauer class and pushing forward to $\text{Spec}(k(X))$ is the same as pairing with the underlying cycle.

But now $(\alpha' - \alpha'_0)(\eta)$ is just the class of $\alpha - \alpha_0$, restricted to $\text{Br}(k(X))$. Hence $\alpha - \alpha_0 = 0$ in $\text{Br}(k(X))$, and since the map $\text{Br}(X) \rightarrow \text{Br}(k(X))$ is injective, we have that $\alpha - \alpha_0 = 0$ in $\text{Br}(X)$, i.e., α comes from $\text{Br}(k)$.

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