Period-index bounds for arithmetic threefolds

BENJAMIN ANTIEAU, ASHER AUEL, COLIN INGALLS, DANIEL KRASHEN, AND MAX LIEBLICH

Abstract. The standard period-index conjecture for Brauer groups of p-adic surfaces S predicts that $\operatorname{ind}(\alpha) | \operatorname{per}(\alpha)^3$ for every $\alpha \in \operatorname{Br}(\mathbf{Q}_p(S))$. Using Gabber's theory of prime-to-ℓ alterations and the deformation theory of twisted sheaves, we prove that $\operatorname{ind}(\alpha) | \operatorname{per}(\alpha)^4$ for α of period prime to $6p$, giving the first uniform period-index bounds over such fields.

Contents

1. Introduction

The purpose of this paper is to prove the following result concerning the period-index problem for the Brauer group.

Theorem 1.1. *Let R be an excellent henselian discrete valuation ring with residue field k of characteristic $p \geq 0$ and with fraction field K. Suppose k is semi-finite or separably closed. Let L be an extension of K of transcendence degree 2, and let $\alpha \in \operatorname{Br}(L)$ be a Brauer class. If α has period prime to p, then*

$$\operatorname{ind}(\alpha) \mid \operatorname{per}(\alpha)^5.$$

If α has period prime to $6p$, then

$$\operatorname{ind}(\alpha) \mid \operatorname{per}(\alpha)^4.$$

Recall from [38] that k is a semi-finite field if it is perfect and if for every prime ℓ, the maximal prime-to-ℓ extension of k is pseudo-algebraically closed with Galois group $\mathbf{Z}_\ell$. Finite fields and pseudo-finite fields are semi-finite. As a special case, we obtain the following result.

Corollary 1.2. *Let S be a geometrically integral surface over a p-adic field K. If $\alpha \in \operatorname{Br}(K(S))$ has period relatively prime to $6p$, then*

$$\operatorname{ind}(\alpha) \mid \operatorname{per}(\alpha)^4.$$

The period of a Brauer class α is its order in the Brauer group and its index is the degree of a division algebra in the Brauer class. The period divides the index and both numbers have the same prime factors. Results bounding the index in terms of the period have motivated many of the developments in the theory of the Brauer group since the beginning of the 20th century. See [4, Section 4] for a survey of results of this type.

For local and global fields, the index equals the period by Albert, Brauer, Hasse, and Noether (see [20, Remark 6.5.6]). For a finitely generated field of transcendence degree 2 over an algebraically closed field, the index equals the period by de Jong [16] (see also [36]). More generally, Artin conjectured that the index equals the period for every C_2 field, and he proved this Brauer classes of period a power of 2 or 3, see [3]. For a field of transcendence degree 1 over a local field, the index divides the square of the period by Saltman [44] for Brauer classes of period prime to the characteristic and Parimala and Suresh [41] in general. Analogous results for fields of transcendence degree 1 over higher local fields are established in [37] and subsequently in [25] by other methods. For fields of transcendence degree 2 over a finite field, the index divides the square of the period by [38].

Such results support the following conjecture (see [10, Section 2.4]).

Conjecture 1.3 (Period-index conjecture)**.** *Let k be an algebraically closed, C_1, or p-adic field, and set $e = 0, 1, 2$ accordingly. Let K be a field of transcendence degree n over a field k. For every $\alpha \in \mathrm{Br}(K)$, we have*

$$\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^{n-1+e}.$$

Based on this conjecture, we do not expect the period-index bound we achieve in Theorem 1.1 to be optimal. However, this is the first proof of a general period-index bound that is uniform in the period for fields of transcendence degree 2 over a local field. For classes of period a power of 2, bounds on the u-invariant are known to imply uniform bounds for the index in terms of the period; our bounds are still better than what can be attained using known u-invariant results for function fields over p-adic fields [34]. There are also nonuniform period-index bounds for C_i fields due to Matzri [39].

Our approach follows a strategy inspired by Saltman [44]: split the ramification of the Brauer class by a field extension of controlled degree and then use geometry to study the unramified Brauer class on a regular proper model. For the former, we draw on, and expand upon, a development due to Pirutka [43] (see Section 2). After splitting the ramification and using Gabber's refined theory of ℓ'-alterations to reduce to a regular (quasi-semistable) model, we reduce the proof of Theorem 1.1 to the following general result. Given an integral scheme X, we write $\kappa(X)$ for its function field; given $\alpha \in \mathrm{H}^2(\kappa(X), \boldsymbol{\mu}_n)$, we write $\mathrm{per}(\alpha)$ and $\mathrm{ind}(\alpha)$ for the period and index of the associated class in $\mathrm{Br}(\kappa(X))$.

Theorem 1.4. *Let R be an excellent henselian discrete valuation ring with residue field k of characteristic $p \geq 0$ and with fraction field K. Suppose that X is a connected regular 3-dimensional scheme, flat and proper over $\mathrm{Spec}\, R$. Let $\alpha \in \mathrm{H}^2(X, \boldsymbol{\mu}_n)$ where n is prime to p. Assume that the Brauer class of α is trivial on all proper closed subschemes of the reduced special fiber $X_{0,\mathrm{red}}$ of dimension at most 1. If $\mathrm{ind}(\alpha|_{\kappa(X_i)}) = \mathrm{per}(\alpha|_{\kappa(X_i)})$ for all irreducible components X_i of $X_{0,\mathrm{red}}$, then $\mathrm{ind}(\alpha_{\kappa(X)}) = \mathrm{per}(\alpha_{\kappa(X)})$.*

Note that the hypothesis, that the Brauer class of α is trivial on all proper subschemes of $X_{0,\mathrm{red}}$ of dimension at most 1, is automatically satisfied if k is semi-finite or separably closed (see Lemma 4.1).

The proof of Theorem 1.4 uses the deformation theory of twisted sheaves to reduce the computation of the index of a Brauer class on a regular model to the existence of twisted sheaves of a certain rank on the reduced special fiber, which we can assume is a strict normal crossings surface. In the case when the special fiber is smooth, this approach was carried out in [36, Proposition 4.3.3.1]. In the general case, we end up proving a version of de Jong and Lieblich's period-index theorems for strict normal crossings surfaces over separably closed and semi-finite fields, respectively.

It is known that Saltman's theorem is the best possible for p-adic curves. Indeed, examples were given by Jacob and Tignol in an appendix to [44] to this effect. Conjecture 1.3 predicts that for a surface over $\mathbf{C}((t))$ one has $\operatorname{ind}(\alpha)|\operatorname{per}(\alpha)^2$, while for a surface over a p-adic field one has $\operatorname{ind}(\alpha)|\operatorname{per}(\alpha)^3$. The nonoptimality of our results is undoubtedly due to our overly generous splitting of ramification. The approach taken in [38] improves these kinds of bounds at the expense of a layer of stacky complexity.

Outline. In Section 2, we generalize work of Pirutka [43] on splitting the ramification of Brauer classes. Section 3 considers Gabber's refined theory of ℓ'-alterations in the context of splitting ramification. Sections 4 and 5 discuss the existence and deformation theory of twisted sheaves on proper models of the function field we consider. Theorems 1.1 and 1.4 are proved in Section 6. Starting in Section 4, we freely use the theory of twisted sheaves. An introduction to the use of twisted sheaves to study questions about the Brauer group can be found in [35] and [36].

Notation. If X is a scheme and R is a commutative ring, we denote by $\mathrm{H}^i(X, \boldsymbol{\mu}_n)$, $\mathrm{H}^i(R, \mathbf{G}_m)$, and so on the corresponding étale cohomology groups. Given a locally noetherian scheme X and a $\mathbf{G}$-gerbe $\pi : \mathscr{X} \to X$ for some closed subgroup $\mathbf{G} \hookrightarrow \mathbf{G}_m$, we will write $\mathrm{Coh}^{(1)}(\mathscr{X})$ for the category of coherent $\mathscr{X}$-twisted sheaves. Similarly, given a locally noetherian scheme X, we will write $\mathrm{Coh}(X)$ for the usual categories of coherent sheaves on X. When F is an $\mathscr{X}$-twisted sheaf and M is an $\mathscr{O}_X$-module, for simplicity we write $F \otimes M$ for the $\mathscr{X}$-twisted sheaf $F \otimes_{\mathscr{O}_{\mathscr{X}}} \pi^* M$.

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2. Splitting ramification

The results of this section are, for the most part, a generalization and reworking of the results of Pirutka [43] (which themselves live in a tradition of ramification-splitting results due to Saltman [44]). We follow Pirutka's strategy with minor modifications so that it works in mixed characteristic, and we give a somewhat different argument on the existence of rational functions whose roots split ramification.

Our ultimate goal in this section is to show that we can split all of the ramification occurring in the Brauer classes of interest to us with relatively small extensions.

Let X be a regular noetherian integral scheme with function field F. Restriction to the generic point induces an injective map $\mathrm{H}^2(X, \mathbf{G}_m) \to \mathrm{H}^2(F, \mathbf{G}_m) \cong \mathrm{Br}(F)$, see [23, Proposition 1.8]. For A a commutative ring, the canonical map $\mathrm{Br}(A) \to \mathrm{H}^2(A, \mathbf{G}_m)_{\mathrm{tors}}$ is an isomorphism; see Hoobler [26]. More generally, if X admits an ample invertible sheaf, then $\mathrm{Br}(X) \to \mathrm{H}^2(X, \mathbf{G}_m)_{\mathrm{tors}}$ is an isomorphism; see [15].

2.1. Ramification.

In this section we fix a ring R and a field F containing R.

Definition 2.1.1. Fix a class $\alpha \in \mathrm{Br}(F)$.

(1) The class α is *unramified* at a discrete valuation ring A of F if α is in the image of the restriction map $\mathrm{Br}(A) \to \mathrm{Br}(F)$. Otherwise, we say that α is *ramified* at A.

(2) The class α is *unramified over* R if α is unramified at every discrete valuation ring A of F such that $R \subset A$.
(3) If X is a regular noetherian integral scheme with function field F and $x \in X$ a point of codimension 1, we say that α is *unramified at* x if α is unramified at the discrete valuation ring $\mathscr{O}_{X,x}$ of F. When additionally $X = \operatorname{Spec} R$ and $x \in R$ is a nonzero divisor, we say that α is unramified at x if it is unramified at the prime ideal (x). In this circumstance, the Weil divisor consisting of the sum of all codimension 1 points of X over which α is ramified is called the *(reduced) ramification divisor* of α.

Remark 2.1.2. Similarly, for a positive integer ℓ invertible in F, we can consider the ramification of classes in $\mathrm{H}^i(F, \boldsymbol{\mu}_\ell^{\otimes j})$ at any discrete valuation ring A of F whose residue field κ has characteristic not dividing ℓ. In this case, $\alpha \in \mathrm{H}^i(F, \boldsymbol{\mu}_n^{\otimes j})$ is unramified if and only if α is contained in the kernel of the residue map $\mathrm{H}^i(F, \boldsymbol{\mu}_n^{\otimes j}) \to \mathrm{H}^{i-1}(\kappa, \boldsymbol{\mu}_n^{\otimes j-1})$ defined in terms of Galois cohomology, see [9, §3.6]. Important cases are $\mathrm{H}^2(F, \boldsymbol{\mu}_\ell)$ and $\mathrm{H}^i(F, \boldsymbol{\mu}_\ell^{\otimes i})$, which correspond to Brauer classes of period ℓ and symbols of length i in Galois cohomology.

Lemma 2.1.3. *Suppose L/F is a finite field extension. If $\alpha \in \mathrm{Br}(F)$ is unramified over R then the restriction $\alpha_L \in \mathrm{Br}(L)$ is unramified over R.*

Proof. For any discrete valuation ring A with fraction field L, the intersection $A \cap F$ is a discrete valuation ring with fraction field F, and if α is unramified at $A \cap F$, then the restriction $\alpha_L \in \mathrm{Br}(L)$ is unramified at A. □

Example 2.1.4. If X is a regular integral scheme with function field F, which admits a proper surjective morphism $X \to \operatorname{Spec} R$, and $\alpha \in \mathrm{Br}(X)$, then by the valuative criterion the image of α under the map $\mathrm{Br}(X) \to \mathrm{Br}(F)$ is unramified over R. Conversely, by purity for regular local rings ([19, Theorem 2'] for schemes of dimension at most 3 and [8] in general), any $\alpha \in \mathrm{Br}(F)$ that is unramified over R is in the image of the map $\mathrm{Br}(X) \to \mathrm{Br}(F)$.

The following gives a useful criterion for checking that Brauer classes become unramified after a finite extension.

Lemma 2.1.5. *Let R be a commutative ring and X a regular integral scheme with a proper surjective morphism $X \to \operatorname{Spec} R$. Let F be the function field of X and L/F a finite extension. Let $\alpha \in \mathrm{Br}(F)$. If for every point $x \in X$ with $R \subset \mathscr{O}_{X,x}$, there exists a regular ring $S \subset L$ that is an integral extension of $\mathscr{O}_{X,x}$, such that the image of α in L is unramified over S, then the image of α in $\mathrm{Br}(L)$ is unramified over R.*

Proof. Let A be a discrete valuation ring with fraction field L containing R. Then the intersection $A \cap F$ is a discrete valuation ring with fraction field F containing R. By the valuative criterion for properness, there exists an R-morphism $\operatorname{Spec} A \cap F \to X$. Then the image $x \in X$ of the closed point is regular with $R \subset \mathscr{O}_{X,x}$. By our hypothesis, there exists a regular ring $S \subset L$ that is an integral extension of the regular local ring $\mathscr{O}_{X,x}$ on which the image of α in $\mathrm{Br}(L)$ is unramified over S. Since A is integral over $A \cap F$ and $\mathscr{O}_{X,x} \subset A \cap F$, it follows that the integral closure of $\mathscr{O}_{X,x}$ in L is contained in A. Hence S, being integral over $\mathscr{O}_{X,x}$, is contained in A. Since α_L is unramified over S, it is unramified at A by definition. □

2.2. **Local description of ramification.** Recall that a regular system of parameters in a regular local ring is a minimal generating set of the maximal ideal. A subsequence of a regular system of parameters is called a *partial* regular system of parameters. Not every regular sequence is a partial regular system of parameters. We fix a positive integer ℓ.

Definition 2.2.1. Let R be a regular local ring with fraction field F and assume that ℓ is invertible in R. We say that $\alpha \in \mathrm{H}^2(F, \boldsymbol{\mu}_\ell^{\otimes 2})$ is *nicely ramified* if α is ramified only along a partial regular system of parameters $x_1, \ldots, x_h$ of R and we can write

$$\alpha = \alpha_0 + \sum_{i=1}^{h}(u_i, x_i) + \sum_{1 \le i < j \le h} m_{i,j}(x_i, x_j)$$

for an unramified class α_0 and some $u_i \in R^\times$ and $m_{i,j} \in \mathbf{Z}$.

More generally, if X is a regular noetherian integral scheme with function field F with ℓ invertible on X, and $\alpha \in \mathrm{H}^2(F, \boldsymbol{\mu}_\ell^{\otimes 2})$, then we say that α is nicely ramified on X if it is nicely ramified at every local ring of X.

We will need the following result, proved in the two-dimensional case in [44] and in the equicharacteristic case in [43, Section 3, Lemma 2].

Lemma 2.2.2. *Let X be a regular noetherian integral scheme with function field F and let $\alpha \in \mathrm{H}^2(F, \boldsymbol{\mu}_\ell^{\otimes 2})$ where ℓ is invertible on X. If α is ramified only along a strict normal crossings divisor, then α is nicely ramified on X.*

Proof. Let R be a local ring of X. By hypothesis, α is ramified only along a partial regular system of parameters $x_1, \dots, x_h$. We proceed by induction on h. For $h = 1$, let F_1 be the fraction field of $R/(x_1)$ and let $\beta = (b) \in F_1^\times / F_1^{\times \ell} = \mathrm{H}^1(F_1, \boldsymbol{\mu}_\ell)$ be the residue of α, where (b) denotes the class (or symbol) of b in $F_1^\times / F_1^{\times \ell}$. Then it follows that β is unramified by considering the Gersten complex

$$\mathrm{H}^2(F, \boldsymbol{\mu}_\ell^{\otimes 2}) \to \bigoplus_{p \in \mathrm{Spec}(R)^{(1)}} \mathrm{H}^1(k(p), \boldsymbol{\mu}_\ell) \to \bigoplus_{q \in \mathrm{Spec}(R)^{(2)}} \mathrm{H}^0(k(q), \mathbf{Z}/\ell\mathbf{Z}).$$

For the construction of the complex in this generality, see [30, Section 1]. Since $R/(x_1)$ is a regular local ring, it is a UFD, and we may write $b = \overline{u}_1 \prod \pi_i^{e_i}$ for irreducible elements π_i and a unit $\overline{u}_1$ of $R/(x_1)$. Since the residue of b at the prime (π_i) is the class of e_i modulo ℓ, it follows that each e_i is a multiple of ℓ, and thus $\beta = (\overline{u}_1) \in F_1^\times / F_1^{\times \ell}$.

Lifting $\overline{u}_1$ to a unit u_1 of R, it follows that the class $\alpha - (u_1, x_1)$ is unramified on R. In particular, we may write $\alpha = \alpha_0 + (u_1, x_1)$ for α_0 unramified as claimed. For $h > 1$, let $\beta \in F_1^\times / F_1^{\times \ell}$ be the residue of α at the prime (x_1), as before. Considering the Gersten complex, the residue of β must be canceled by the residues of α along primes in $R/(x_1)$. In particular, it follows that β can only be ramified along the primes $\overline{x}_2, \dots, \overline{x}_h$ in $R/(x_1)$. Since $R/(x_1)$ is a regular local ring, it is a UFD, and we may represent β by an element $\overline{b} = \overline{u}_1 \prod_{i=2}^h \overline{x}_i^{m_{i,1}}$ with $\overline{u}_1$ a unit in $R/(x_1)$. In particular, we can lift $\overline{b}$ to $b = u_1 \prod_{i=2}^h x_i^{m_{i,1}}$, where $u_1 \in R^\times$. It follows that

$$\alpha - (b, x_1) = \alpha - (u_1, x_1) - \sum_{i=2}^h m_{i,1}(x_i, x_1)$$

is unramified along (x_1) and only ramified along the primes (x_i) for $i = 2, \dots, h$. By induction, we may write

$$\alpha - (b, x_1) = \alpha_0 + \sum_{j=2}^h (u_j, x_j) + \sum_{j,k \neq 1} m_{j,k}(x_j, x_k),$$

yielding $\alpha = \alpha_0 + \sum (u_i, x_i) + \sum m_{i,j}(x_j, x_k)$ as desired. □

2.3. **Putting ramification in nice position.** We will need the following generalization of [43, Lemma 3], which from the toroidal geometry perspective, is related to the process of barycentric subdivision. The standard reference for toroidal geometry is [31], which is written over an algebraically closed base field. However, all the constructions work over an arbitrary base scheme as outlined in [18, IV Remark 2.6].

Definition 2.3.1. Let $D \subset X$ be a strict normal crossings divisor in a regular noetherian scheme. We define a presentation of D to be a finite collection $\{D_i\}_{i \in I}$ of regular, but not necessarily connected, divisors such that $D = \cup_{i \in I} D_i$ and $D_i = D_j$ implies $i = j$. We call $|I|$ the length of the presentation.

For example, we may choose our presentation to simply consist of the irreducible components of D. On the other hand, the next lemma shows that after possibly blowing up, we may find a presentation whose length is bounded by the dimension of X.

Lemma 2.3.2. *Let X be a regular noetherian scheme of dimension d and suppose that $D \subset X$ is a strict normal crossings divisor. There exists a sequence of blowups along regular subschemes $f : X' \to X$ such that $f^{-1}(D)$ admits a presentation of length at most d.*

Before giving the proof, we recall some combinatorial notions related to inverse images of strict normal crossings divisors under certain blowups.

Definition 2.3.3. An *(abstract) simplicial complex* Σ is a collection of nonempty finite sets, called *simplices*, closed under inclusion. The union of all simplices is the *vertex set* of Σ.

(1) The elements $\sigma \in \Sigma$ of cardinality $i+1$ are called *i-simplices*. We write Σ_i for the subset of all i-simplices in Σ. By abuse of notation, we also use the term vertex for a 0-simplex and the symbol Σ_0 for the vertex set.
(2) The *dimension* of Σ is defined to be the maximal $i \geq 0$ such that $\Sigma_i \neq \varnothing$, assuming that this maximum exists.
(3) Given a simplicial complex and a non-empty simplex $\sigma \subset \Sigma$, we define the *star subdivision* $\Sigma \star \sigma$ with respect to σ to be the simplicial complex whose vertex set is the vertex set of Σ together with a new vertex e_σ, and whose simplices are

$$\{\tau \in \Sigma \ : \ \sigma \not\subseteq \tau\} \bigcup \{(\tau \smallsetminus J) \cup \{e_\sigma\} \ : \ \emptyset \neq J \subseteq \sigma \subseteq \tau \in \Sigma\}$$
$$= \{\tau \in \Sigma \ : \ \sigma \not\subseteq \tau\} \bigcup \{(\tau' \cup \{e_\sigma\} \ : \ \emptyset \neq J \subseteq \sigma \subseteq \tau' \cup J \in \Sigma, \text{ some } J \subseteq \Sigma_0 \setminus \tau'\}.$$

(4) We formally define $\Sigma \star \varnothing = \Sigma$.

Remark 2.3.4. Let Σ be a simplicial complex.

(1) If σ is a 0-simplex, then $\Sigma \star \sigma$ is isomorphic to Σ by sending e_σ to σ.
(ii) If σ and τ are simplices such that neither $\sigma \subseteq \tau$ nor $\tau \subseteq \sigma$, then $(\Sigma \star \sigma) \star \tau = (\Sigma \star \tau) \star \sigma$. In particular, for any subset $\Sigma' \subseteq \Sigma$ consisting of simplices none of which contain any other, we can define the iterated star subdivision $\Sigma \star \Sigma'$ with respect to all simplices in Σ'.

Definition 2.3.5. Let Σ be a simplicial complex.

(a) The *barycentric subdivision* $\mathrm{Sd}(\Sigma)$ of Σ is the iterated star subdivision
$$(((\Sigma \star \Sigma_d) \star \Sigma_{d-1}) \star \cdots) \star \Sigma_1,$$
see [31, III, §2A].
(b) The *order complex* $\mathrm{Fl}(\Sigma)$ of Σ is the simplicial complex with a vertex for each simplex of Σ and whose i-simplices are all length $i+1$ flags of inclusions of simplices of Σ.

We will need the following combinatorial lemma.

Lemma 2.3.6. *There is a natural isomorphism of simplicial complexes* $\mathrm{Sd}(\Sigma) \cong \mathrm{Fl}(\Sigma)$.

Proof. See [33, §2.1.5]. □

Now, we apply these definitions to a simplicial complex associated to a presentation of a strict normal crossings divisor.

Definition 2.3.7. Let $D \subset X$ be a strict normal crossings divisor in a regular noetherian scheme and let $\{D_i\}_{i \in I}$ be a presentation of D. We define a simplicial complex $\Sigma(D) = \Sigma(D, \{D_i\}_{i \in I})$ with vertex set I such that a subset $J \subset I$ is in $\Sigma(D)$ whenever $\cap_{j \in J} D_j \neq \emptyset$. We call $\Sigma(D)$ the *naive dual complex* associated to the presentation.

Remark 2.3.8. The usual dual complex, which includes distinct i-simplices for each irreducible component of each intersection of $i+1$ components of D, has better homotopical properties (e.g., see [42]); the naive version will suffice for our purposes.

Following standard conventions, for $J \subset I$, we will let D_J denote the intersection $\cap_{j \in J} D_j$. By hypothesis, D_J is nonempty whenever $J \in \Sigma(D)$ and since D is snc is a regular subscheme of X of codimension $|J|$.

Remark 2.3.9. The dimension of $\Sigma(D)$ is bounded above by the dimension of X.

An important fact relating the geometry of the pair (X, D) to the combinatorics of the dual complex $\Sigma(D)$ is the following.

Lemma 2.3.10. *Let $\{D_i\}_{i\in I}$ be a presentation of a snc divisor D in a regular noetherian scheme X. Let σ be an i-simplex of $\Sigma(D)$ and let $f\colon \mathrm{Bl}_{D_\sigma} X \to X$ be the blowup of X along D_σ. We let*

- *$\widetilde{D} = f^{-1}(D)$ denote the inverse image of D,*
- *$\widetilde{D}_i = f^{\mathrm{st}}(D_i)$ denote the strict transform of D_i for $i \in I$,*
- *and E denote the exceptional divisor of f.*

The naive dual complex $\Sigma(\widetilde{D})$ with respect to the presentation $\{E\} \sqcup \{\widetilde{D}_i\}_{i\in I}$ of $\widetilde{D}$ is naturally isomorphic to $\Sigma(D) \star \sigma$, where the new vertex e_σ corresponds to E.

Note that an analogous statement is made in [13, Proposition 3.3.15] in the case of the usual dual complex.

Proof. We use the natural bijection of vertex sets $\{\Sigma(D) \star \sigma\}_0 \cong \Sigma(\widetilde{D})_0$ which is equality on $I = \Sigma(D)_0$ and sends e_σ to E. To prove the lemma, we need to show that the incidence of the divisors E and $\widetilde{D}_i$ satisfy the same incidence relations as the 0-simplices of the complex $\Sigma(D) \star \sigma$, namely that

(1) for $J \subseteq I$, $\widetilde{D}_J \neq \emptyset$ if and only if $D_J \neq \emptyset$ and $\sigma \not\subseteq J$;
(2) for $J = J' \sqcup \{e_\sigma\}$ where $J' \subset I$, $\widetilde{D}_J \neq \emptyset$ if and only if $D_{J'\sqcup J''} \neq \emptyset$ for some $J'' \subseteq \sigma \subseteq J' \cup J''$ with $\emptyset \neq J'' \subseteq I \setminus J'$.

Geometrically, the first condition says that $\cap_{i\in J}\widetilde{D}_i$ will be nonempty if and only if $\cap_{i\in J} D_i$ is nonempty and is not contained in D_σ. The only nontrivial part of the statement then is the assertion that if the intersection D_J is nonempty and is contained in D_σ, then $\widetilde{D}_J$ is empty. To see this, note that $\widetilde{D}_J \subset \widetilde{D}_\sigma$, so it suffices to assume that $J = \sigma$. Looking étale locally near D_σ, we can replace X with $\operatorname{Spec} R$ for a regular local ring R, and where $D_i, i \in \sigma$ is cut out by $x_1, \ldots x_m$, which form part of a regular system of parameters. The blowup is then given as the relative proj of the graded ring $R[t_1, \ldots, t_m]/(x_i t_j - x_j t_i)$ for pairs of indices $i \neq j$, with the strict transform $\widetilde{D}_i$ cut out by the ideal (t_i, x_i). It follows that the stratum $\widetilde{D}_\sigma$ is defined by an ideal which contains the irrelevant ideal, and is therefore empty.

The content of the second part is the statement that a stratum $\widetilde{D}_{J'}$, where $J' \not\subseteq \sigma$ (this condition is ensured by $J'' \subset \sigma \setminus J'$) should have a nonempty intersection with the exceptional divisor E exactly when the stratum $D_{J'}$ has a nontrivial intersection with a stratum $D_{J''}$ with mutual intersection $D_{J'\sqcup J''}$ contained in the blowup locus D_σ.

Suppose that J' and J'' are chosen as above. As $\sigma \not\subseteq J'$ (since the elements of J'' are contained in σ but not in J'), $D_{J'} \not\subseteq D_\sigma$. On the other hand $D_\sigma \cap D_{J'} \supseteq D_{J''} \cap D_{J'} = D_{J''\sqcup J'} \neq \emptyset$, and hence D'_J nontrivially intersects the locus D_σ which is being blown up. But therefore $\widetilde{D}_{J'}$, which may be identified with the strict transform of $D_{J'}$ nontrivially intersects E, as desired.

In the other direction, if for some $J' \not\subseteq \sigma$, all such intersections were trivial, then it would follow that $D_{J'\sqcup\sigma}$ is also trivial, showing that $D_{J'}$ is disjoint from D_σ. But in this case, it is easy to see that $D_{J'}$ will be disjoint from E as claimed. □

Now, we can use Lemma 2.3.10 to prove Lemma 2.3.2.

Proof of Lemma 2.3.2. Consider the proper birational morphism $f : X' \to X$ obtained by sequentially blowing up the strata of D, first blowing up the 0-dimensional strata of D, then the strict transforms of the 1-dimensional strata of D, then the strict transforms of the 2-dimensional strata of D, etc. The inverse image $f^{-1}(D)$ is a strict normal crossings divisor in X' and, by Lemma 2.3.10, its naive dual complex is the barycentric subdivision of $\Sigma(D)$. Given its interpretation in Lemma 2.3.6 as the order complex of $\Sigma(D)$, the vertices of the barycentric subdivision can be colored with at most d colors "by dimension", with the subset of vertices corresponding to elements in $\Sigma(D)_i$ having color i. This coloring has the property that no two distinct vertices of the same color are both contained in a simplex; equivalently, the correspondingly colored components of $f^{-1}(D)$ are disjoint in X'. Hence the union of all irreducible components of the same color is a regular divisor. Therefore, we can express $f^{-1}(D)$ as the union of at most d regular (but not necessarily connected) divisors, as required. If need be, we can further blow up smooth points to get a union of exactly d regular divisors. □

2.4. Construction of rational functions for splitting ramification.

Notation 2.4.1. Let $\{V_i\}_{i\in I}$ be a family of cycles on X. Given a subset $J \subset I$, let V_J denote the naive intersection cycle, defined as follows. Given two integral subschemes A and B of X, the naive intersection cycle is $(A\cap B)_{\mathrm{red}}$, written as a sum of its irreducible components. Given two cycles $\sum a_i W_i$ and $\sum b_j W_j$, the naive intersection is the union of the naive intersections $W_i \cap W_j$. (This is purely a way of measuring dimensions of support as a notational convenience, nothing else. We thus ignore coefficients and intersection multiplicities.)

Definition 2.4.2. A collection of irreducible subschemes $\{W_i\}_{i\in I}$ of X *intersects properly* if for every subset $J \subset I$ we have $\operatorname{codim} W_J \geq \sum_{i\in J} \operatorname{codim} W_i$ (using the convention that $\operatorname{codim} \emptyset = \infty$). A collection of cycles $\{V_i\}_{i\in I}$ on X intersect properly if any collection $\{W_i\}_{i\in I}$, where W_i is an irreducible component of the support of V_i for all $i \in I$, intersects properly.

Lemma 2.4.3. *Let X be a scheme. Suppose that $\{W_i\}_{i\in I}$ is a collection of cycles of X that intersect properly. If $W \subset X$ is an irreducible subscheme such that, for every subset $J \subset I$, the scheme W intersects each irreducible component of W_J properly, then $\{W_i\} \cup \{W\}$ intersects properly.*

Proof. We omit the proof. □

Now, we prove a lemma which is a direct generalization of the lemma in the correction to [44], and which can be viewed as a version of Kawamata's trick, see [1, §5.3].

Lemma 2.4.4. *Let X be a regular scheme admitting an ample invertible sheaf and let $D_1, \ldots, D_d$ be a collection of regular divisors on X whose union is a strict normal crossings divisor. Let*

$$\widetilde{D}_i = \sum_{j=1}^{d} m_{i,j} D_j, \quad i = 1, \ldots, n$$

be a collection of integer linear combinations of the D_j. In this case, for $i = 1, \ldots, n$, there exist rational functions f_i and divisors E_i on X such that

(1) $\operatorname{div}(f_i) = \widetilde{D}_i + E_i$, *and*
(2) *the collection $\{E_i\}_{i=1,\ldots,n} \cup \{D_j\}_{j=1,\ldots,d}$ intersects properly.*

Proof. We construct the f_i inductively.

Base case $i = 1$. Let P be a (scheme-theoretic) disjoint union of closed points, with one closed point in each irreducible component of D_I for each subset $I \subset \{1, \ldots, d\}$. Let R be the semilocal ring of the points of P (which exists since we can put P in a single quasi-affine open of X, using the ample invertible sheaf [21, Théorème 4.5.2] and the graded prime-avoidance lemma [17, Section 3.2]). By [7, Chapitre II, Section 5.4, Proposition 5], since finitely generated projective modules over R are free, it follows that each D_i is principal on R. In particular, we may write D_i as the zero locus of a function $x_i \in R$. We then have, upon setting $f_1 = \prod x_j^{m_{1,j}}$, that $(f_1) = \widetilde{D}_1 + E_1$ and the support of E_1 contains none of the strata D_I. In particular, since E_1 is a divisor, the codimension of $E_1 \cap D_I$ in D_I is at least 1 as desired.

Induction step. Suppose we have previously defined $f_1, \ldots, f_{r-1}$ so that $\operatorname{div}(f_i) = \widetilde{D}_i + E_i$ with $\{E_i\}_{i=1,\ldots,r-1} \cup \{D_j\}_{j=1,\ldots,d}$ intersecting properly. For every pair of subsets $I \subset \{1, \ldots, d\}$ and $J \subset \{1, \ldots, r-1\}$, consider the intersection $D_I \cap E_J$ and let P be a scheme theoretic union consisting of at least one closed point from each irreducible component of each of these nonempty intersections for every I, J as above. Let R be the semilocal ring at P. As above, we may write D_i as the zero locus of some function $x_i \in R$ on $\operatorname{Spec} R \subset X$, and considering the x_i as rational functions on X, we set $f_r = \prod x_j^{m_{r,j}}$. It follows that $(f_r) = \widetilde{D}_r + E_r$ where the support of E_r contains no irreducible component of the strata $D_I \cap E_J$, with $J \subset \{1, \dots, r-1\}$. Therefore $\{E_i\}_{i=1,\dots,r} \cup \{D_j\}_{j=1,\dots,d}$ intersects properly as desired. □

Notation 2.4.5. Let $T \in \mathrm{Mat}_{n,d}(\mathbf{Z})$ be an $n \times d$ matrix. For subsets $I \subset \{1, \dots n\}$ and $J \subset \{1, \dots, d\}$, let $T_{I,J}$ denote the $|I| \times |J|$-submatrix of T with rows corresponding to the elements of I and columns corresponding to the elements of J.

Definition 2.4.6. Let ℓ be a prime and n, d be positive integers with $n \geq d$. An $n \times d$ matrix $T \in \mathrm{Mat}_{n,d}(\mathbf{Z})$ is *ℓ-Pirutka* if for all nonempty subsets $I \subset \{1, \dots, n\}$ and $J \subset \{1, \dots, d\}$, with $|I| - |J| = n - d$, the submatrix $T_{I,J}$ has (maximal) rank $|J|$ modulo ℓ.

Lemma 2.4.7. *Let R be a regular local ring with fraction field F and $\alpha \in \mathrm{H}^t(F, \boldsymbol{\mu}_\ell^{\otimes t})$ where ℓ is invertible in R. Let $x_1, \dots, x_n \in R$ be a regular system of parameters and suppose that $\alpha = (u_1, \dots, u_{t-h}, x_1, \dots, x_h)$ with u_i units in R. If $L = F(\sqrt[\ell]{x_1}, \dots, \sqrt[\ell]{x_h})$ and S is the integral closure of R in L, then*

(1) *S is a regular local ring with maximal ideal generated by $\sqrt[\ell]{x_1}, \dots, \sqrt[\ell]{x_h}, x_{h+1}, \dots, x_n$,*
(2) *the class α_L has trivial residue at each codimension one prime of S.*

Proof. We omit the proof that $S = R[z_1, \dots, z_h]/(z_1^\ell - x_1, \dots, z_h^\ell - x_h)$ and is regular with maximal ideal $\mathfrak{m} = (z_1, \dots, z_h, x_{h+1}, \dots, x_n)$.

Now, for a prime $\mathfrak{P} \subset R$ of height one and a prime $\mathfrak{Q} \subset S$ lying over it with ramification index e, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^t(F, \boldsymbol{\mu}_\ell^{\otimes t}) & \longrightarrow & \mathrm{H}^{t-1}(R/\mathfrak{P}, \boldsymbol{\mu}_\ell^{\otimes t-1}) \\ \downarrow & & \downarrow {\scriptstyle e\,\mathrm{res}} \\ \mathrm{H}^t(L, \boldsymbol{\mu}_\ell^{\otimes t}) & \longrightarrow & \mathrm{H}^{t-1}(S/\mathfrak{Q}, \boldsymbol{\mu}_\ell^{\otimes t-1}) \end{array}$$

of residue maps, which shows that α_L only ramifies at primes lying over the ramification locus of α. In particular, α_L can only ramify over the primes (z_i) for $i = 1, \dots, h$, which each have ramification index ℓ over (x_i). Since all residues of α are ℓ-torsion, it follows from the diagram above that α_L is unramified. □

Lemma 2.4.8. *Let X be a regular scheme of dimension d admitting an ample invertible sheaf and $D_1, \dots, D_d$ be a collection of regular divisors of X whose union is snc. Let ℓ be a prime invertible on X and let $\alpha \in \mathrm{H}^2(F, \boldsymbol{\mu}_\ell^{\otimes 2})$ be a class ramified only along the union of the D_i. Let $T = (m_{ij})$ be an ℓ-Pirutka $n \times d$ matrix. For each $i = 1, \dots, n$, let $\widetilde{D}_i = \sum_{j=1}^d m_{ij} D_j$ and let f_i and E_i be as in Lemma 2.4.4. If $L = F(\sqrt[\ell]{f_1}, \dots, \sqrt[\ell]{f_n})$, then α_L is unramified.*

Proof. By Lemma 2.2.2, for any point $z \in X$, we have that

$$\alpha = \alpha_0 + \sum (u_i, x_i) + \sum_{i,j} m_{i,j}(x_i, x_j) \tag{2.4.1}$$

for $u_i \in \mathscr{O}_{X,z}^\times$ and x_i local equations for D_i in $\mathscr{O}_{X,z}$.

To show that α becomes unramified in L, by Lemma 2.1.5, it suffices to show that for every point $z \in X$ and each term in (2.4.1), there is a subfield of L where that term becomes unramified over some regular subring contained in L and integral over $\mathscr{O}_{X,z}$. For example, by Lemma 2.4.7, any term of the form (u_i, x_i) will become unramified over a regular subring of an extension $F(\sqrt[\ell]{g_i})$ where g_i is a local equation for D_i at z; any term of the form (x_i, x_j) will become unramified over a regular subring of an extension $F(\sqrt[\ell]{g_i}, \sqrt[\ell]{g_j})$.

We thus seek the following: for each point $z \in X$ and each $j = 1, \dots, d$, an element $g_j \in F$ such that

(1) g_j is a local equation for rD_j at z, where $r \equiv 1 \bmod \ell$;
(2) $F(\sqrt[\ell]{g_j}) \subset L$.

Choose J maximal with respect to inclusion so that $z \in D_J$. If $J = \emptyset$ (so that α is unramified over $\mathscr{O}_{X,z}$), then $g_j = 1$ works for all j. Otherwise, choose any $j_0 \in J$; we will find $g_{j_0} \in F$ satisfying conditions (1) and (2) above.

We claim that we can find $I \subset \{1,\dots n\}$ with $|I| - |J| = n-d$ and $z \notin \cup E_i$. To see this, set

$$I' = \{i \in \{1,\dots,n\} \,|\, z \in E_i\}.$$

Since $z \in E_{I'} \cap D_J$, it follows by the properness of the intersection that $|I'|+|J| \leq d = \dim(X)$. In particular, there are at most $d-|J|$ indices i such that $z \in E_i$. This means we can find a set of $n-(d-|J|)$ indices i such that $z \notin E_i$. Let I be such a set. Since T is an ℓ-Pirutka matrix, the submatrix $T_{I,J}$ has full rank $|J|$ modulo ℓ, and hence we can find $a_i \in \mathbf{Z}$ for $i \in I$ such that $\sum_{i\in I} a_i m_{i,j} \equiv \delta_{j,j_0} \bmod \ell$ for each $j \in J$. Translating in terms of $\widetilde{D}_i$ and D_i, this says that there exists $r \equiv 1 \bmod \ell$ such that

$$\sum_{i\in I} a_i \widetilde{D}_i = rD_{j_0} + \sum_{j\notin J} b_j D_j$$

and therefore

$$\operatorname{div}\left(\prod_{i\in I} f_i^{a_i}\right) = rD_{j_0} + \sum_{j\notin J} b_j D_j + \sum_{i\in I} a_i E_i.$$

Let $g_{j_0} = \prod_{i\in I} f_i^{a_i}$. Since $z \notin E_i$ for all $i \in I$ and $z \notin D_j$ for all $j \notin J$, we find that g_{j_0} is a local equation for rD_{j_0} in $\mathscr{O}_{X,z}$. It is clear that $\sqrt[\ell]{g_{j_0}} \in L$. □

Theorem 2.4.9. *Let X be a regular scheme of dimension d admitting an ample invertible sheaf. Let ℓ be a prime invertible on X and $\alpha \in \mathrm{H}^2(F, \boldsymbol{\mu}_\ell^{\otimes 2})$ be ramified along a strict normal crossings divisor. If there is an ℓ-Pirutka $n \times d$ matrix, then we can find rational functions $f_1,\dots,f_n \in F$ so that α becomes unramified in $L = F(\sqrt[\ell]{f_1},\dots,\sqrt[\ell]{f_n})$.*

Proof. By Lemma 2.3.2 we can perform a sequence of blowups to X so as to make the ramification divisor of α contained in an snc divisor that we can write as a union $D_1 \cup \cdots \cup D_d$ of regular divisors. Now the result is an immediate application of Lemma 2.4.8. □

2.5. **Some Pirutka matrices.**

Example 2.5.1. Pirutka's proof in [43] uses the following ℓ-Pirutka matrix. Consider $n = d^2$, and let T be the $d^2 \times d$ matrix given by d (vertical) copies of the $d \times d$ identity matrix. The condition is now: for every subset of columns J, and subset of rows I of order $d^2 - d + |J|$, we have full rank. But notice that since $|J| \geq 1$, we are always removing fewer than d rows. Since each row of the identity matrix occurs d times, each row of the identity matrix must still occur in the I, J-minor, showing that $T_{I,J}$ has full rank.

Example 2.5.2. The 3×3 matrix

$$\begin{pmatrix} 1 & 3 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

considered in [43, Remark 5] is ℓ-Pirutka for all primes $\ell > 3$.

Example 2.5.3. The 4×3 matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

found in [43, Remark 4] is ℓ-Pirutka for all primes ℓ.

Remark 2.5.4. A computer search shows that

(1) there are no 2-Pirutka 2×2 or larger square matrices, and
(2) there are no 3-Pirutka 3×3 or larger square matrices.

It is easy to make a 2×2 matrix that is ℓ-Pirutka for all primes $\ell > 2$. This allows one to split the ramification of classes of odd prime period on surfaces using roots of two rational functions, which reproduces results of Saltman [45] except for classes of period 2.

Question 2.5.5. For which ℓ, n, d do there exist ℓ-Pirutka $n \times d$ matrices?

It is clear that if $n \gg d$, then there exist ℓ-Pirutka $n \times d$ matrices. We also have the following bound for square matrices.

Proposition 2.5.6. *If $\ell > \binom{2n-1}{n}$ is prime, then there exist ℓ-Pirutka $n \times n$ matrices.*

Proof. First note that an $n \times n$ matrix T is ℓ-Pirutka if and only if all maximal minors of the $n \times 2n$ matrix $A = (I_n | T)$ do not vanish modulo ℓ. We will consider building the matrix $A = (e_1, \ldots, e_n, t_1, \ldots, t_n)$ by inserting the columns t_i one at a time. For inserting the first column, we simply require that all entries in t_1 do not vanish. Once the first column has been fixed, we require that t_2 avoids the $\binom{n+1}{n-1}$ hyperplanes defined by the maximal minors containing t_2, which is certainly possible if $\ell > \binom{n+1}{n-1}$. Similarly, once the first k columns have been fixed, we then require that t_{k+1} avoids the $\binom{n+k}{n-1}$ hyperplanes defined by the maximal minors containing t_{k+1}, which is certainly possible if $\ell > \binom{n+k}{n-1}$. When $k = n-1$, then inserting the final column t_n is certainly possible if the stated bound is satisfied. ☐

Of course this bound is far from sharp. The hyperplanes in the above proof are not in general position.

3. Alterations

In this section, we use Gabber's theory of prime-to-ℓ alterations over a discrete valuation ring; see [28], [29]. For an example of the statement we are interested in, see [11, Théorème 3.25] and its proof.

Definition 3.1. Let ℓ be a prime number and X a scheme of finite type over an excellent ring. An *ℓ'-alteration* $X' \to X$ is a proper surjective generically finite map such that for every maximal point η of X, there exists a maximal point η' of X' over η such that the residue field extension $\kappa(\eta')/\kappa(\eta)$ has degree prime to ℓ.

Lemma 3.2. *Let X be an integral scheme, $X' \to X$ an ℓ'-alteration, η' a maximal point of X' dominating X, and $\alpha \in \mathrm{Br}(\kappa(X))$. If $\mathrm{ind}(\alpha_{\kappa(\eta')}) = \ell^N$ then $\mathrm{ind}(\alpha) = \ell^N$.*

Proof. Because $\kappa(\eta')/\kappa(X)$ has degree prime-to-ℓ, the result follows by a standard restriction-corestriction argument. ☐

Definition 3.3. Let R be a discrete valuation ring, $s \in \operatorname{Spec} R$ the closed point, and X an R-scheme. If X is equidimensional, flat and of finite type over R, the generic fiber of X over $\operatorname{Spec} R$ is smooth, and the reduced special fiber $X_{0,\mathrm{red}}$ is a strict normal crossings divisor on X, then X is said to be *quasi-semistable* over R.

The following two results are a distillation of the main results of Gabber's theory of uniformization by ℓ'-alterations; see [28, Theorem 1.4], [29, X.2], and [11, Théorème 3.25].

Lemma 3.4. *If X is quasi-semistable over R, then $X \to \operatorname{Spec} R$ is étale locally of the form*

$$X = \operatorname{Spec} R[t_1, \ldots, t_n]/(t_1^{a_1} \cdots t_r^{a_r} - \pi), \tag{3.0.1}$$

where π is a uniformizing parameter of R.

Theorem 3.5 (Gabber)**.** *If R is an excellent henselian discrete valuation ring with residue field k of characteristic $p \geq 0$ and fraction field K, and X is a proper scheme over R, then for any prime $\ell \neq p$, there exists a commutative diagram of ℓ'-alterations*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} R' & \longrightarrow & \operatorname{Spec} R \end{array}$$

with R' an excellent henselian discrete valuation ring such that X' is a regular scheme that is quasi-semistable and projective over R'.

Proposition 3.6. *Let R be an excellent henselian discrete valuation ring, X be an integral scheme proper over R of relative dimension d. If $\alpha \in \mathrm{Br}(\kappa(X))[\ell]$, then there exists a diagram of morphisms*

(3.0.2)
$$\begin{array}{ccccccc} Y' & \xrightarrow{h} & Y & \xrightarrow{\quad g \quad} & X' & \xrightarrow{f} & X \\ \downarrow & & & \searrow \quad \swarrow & & & \downarrow \\ \operatorname{Spec} R'' & & \longrightarrow & \operatorname{Spec} R' & \longrightarrow & & \operatorname{Spec} R \end{array}$$

where

1. *R'/R and R''/R' are finite extensions of excellent henselian discrete valuation rings such that R''/R has degree prime to ℓ,*
2. *f and h are ℓ'-alterations,*
3. *X' and Y' are regular and integral,*
4. *$X' \to \operatorname{Spec} R'$ and $Y' \to \operatorname{Spec} R''$ are projective and quasi-semistable,*
5. *Y is integral and the function field extension induced by g has the form*
$$\kappa(Y) = \kappa(X')(\sqrt[\ell]{f_1}, \ldots, \sqrt[\ell]{f_N})$$
for some N, and
6. *$\alpha_{\kappa(Y')}$ lies in the subgroup $\mathrm{Br}(Y')[\ell] \subseteq \mathrm{Br}(\kappa(Y'))[\ell]$.*

Moreover, if there exists an ℓ-Pirutka $n \times (d+1)$ matrix, then we may take $N = n$.

Proof. By Theorem 3.5, there exists a commutative diagram of ℓ'-alterations

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} R_1 & \longrightarrow & \operatorname{Spec} R \end{array}$$

where X_1 is regular and integral, and projective over R_1. Consider the ramification divisor D_1 of $\alpha_{\kappa(X_1)}$ on X_1. By an application of Gabber's embedded uniformization (see [28, Theorem 1.4]), there exists a further commutative diagram of ℓ'-alterations

$$\begin{array}{ccc} X' & \xrightarrow{f_2} & X_1 \\ \downarrow & & \downarrow \\ \operatorname{Spec} R' & \longrightarrow & \operatorname{Spec} R_1 \end{array}$$

where X' is regular and integral, and projective and quasi-semistable over R', and such that $f_2^{-1}(D_1)_{\mathrm{red}} \cup (X_s)_{\mathrm{red}}$ has normal crossings (but not necessarily strict normal crossings), where $s \in \operatorname{Spec} R'$ is the closed point. After blowing up X', we may assume that $f_2^{-1}(D_1)_{\mathrm{red}} \cup (X'_s)_{\mathrm{red}}$ is a strict normal crossings divisor [14, Paragraph 2.4] (see also [12]). On the other hand, the ramification divisor of $\alpha_{\kappa(X')}$ must be contained in $f_2^{-1}(D_1)_{\mathrm{red}}$, so (after this blowing up) that $\alpha_{\kappa(X')}$ has ramification divisor with strict normal crossings. We compose these two squares to arrive at the right-most square in the desired diagram. We can also assume, by possibly taking a further prime-to-ℓ extension, that R' has a primitive ℓth root of unity so that we may apply the results of Section 2.2 to classes in the Brauer group.

By Theorem 2.4.9 and Example 2.5.1, there exist rational functions $f_1, \ldots, f_N$ in $\kappa(X')$ such that α_L is unramified, where $L = \kappa(X')(\sqrt[\ell]{f_1}, \ldots, \sqrt[\ell]{f_N})$. (We can always choose $N = (d+1)^2$, and moreover, if there exists an ℓ-Pirutka $n \times (d+1)$ matrix, we can take $N = n$.) Let Y be the normalization of X' in L and $g : Y \to X'$ the induced map. We now apply Theorem 3.5

again to arrive at an ℓ'-alteration

$$\begin{array}{ccc} Y' & \xrightarrow{h} & Y \\ \downarrow & & \downarrow \\ \operatorname{Spec} R'' & \longrightarrow & \operatorname{Spec} R' \end{array}$$

where Y' is regular and integral, and projective and quasi-semistable over R''. Since $\alpha_{\kappa(Y)}$ is unramified, we have by Lemma 2.1.3 that $\alpha_{\kappa(Y')}$ is unramified, whence $\alpha_{\kappa(Y')} \in \operatorname{Br}(Y')$ by Example 2.1.4. □

4. Existence of twisted sheaves on strict normal crossings surfaces

The purpose of this section is to show that period equals index for Brauer classes on strict normal crossings surfaces. More precisely, if X is a snc surface over a separably closed or semi-finite field and $\alpha \in \mathrm{H}^2(X, \boldsymbol{\mu}_n)$ is a cohomology class with n invertible on X, we show in Proposition 4.8 below that there is an Azumaya algebra A of degree n on X with cohomology class α (equivalently, there exists a twisted sheaf of rank n and trivial determinant).

We note first that Brauer groups of curves vanish in a high degree of generality.

Lemma 4.1. *Let C be a curve over a field k of characteristic $p \geq 0$. If k is separably closed (resp. k is semi-finite and C is proper), then $\operatorname{Br}(C)[n] = 0$ for n prime-to-p (resp. $\operatorname{Br}(C) = 0$).*

Proof. If k is separably closed, then this is [24, Corollaire 1.3]. Thus, assume that k is semi-finite and C is proper over k. Consider the Leray spectral sequence

$$\mathrm{E}_2^{st} = \mathrm{H}^s(k, \mathrm{R}^t\, \pi_* \mathbf{G}_{m,C}) \Longrightarrow \mathrm{H}^{s+t}(C, \mathbf{G}_{m,C})$$

for the structural morphism $\pi : C \to \operatorname{Spec} k$. The only possible contributions to $\mathrm{H}^2(C, \mathbf{G}_{m,C})$ are

$$\begin{aligned} &\mathrm{H}^0(k, \mathrm{R}^2\, \pi_* \mathbf{G}_{m,C}), \\ &\mathrm{H}^1(k, \mathrm{R}^1\, \pi_* \mathbf{G}_{m,C}) \cong \mathrm{H}^1(k, \mathbf{Pic}_{C/k}), \\ &\mathrm{H}^2(k, \mathrm{R}^0\, \pi_* \mathbf{G}_{m,C}) \cong \mathrm{H}^2(k, \mathbf{G}_m). \end{aligned}$$

The last vanishes because k is semi-finite. To analyze the second term, let $\widetilde{C} \to C$ be the normalization of the largest reduced subscheme C_{red} of C. By [6, Corollary 9.2.11], there is an exact sequence

$$0 \to G \to \mathbf{Pic}_{C/k} \to \mathbf{Pic}_{\widetilde{C}/k} \to 0$$

of étale sheaves over k, where G is a connected linear algebraic group. Moreover, since $\widetilde{C}/k$ is smooth (since k is perfect), there is (e.g., [6, Propositions 9.2.3, 9.2.14]) an exact sequence

$$0 \to \mathbf{Pic}^0_{\widetilde{C}/k} \to \mathbf{Pic}_{\widetilde{C}/k} \to A \to 0,$$

where $\mathbf{Pic}^0_{\widetilde{C}/k}$ is an abelian variety and A is an étale sheaf on k with $A(k^s) \cong \mathbf{Z}^r$, where r is the number of irreducible components of C. In fact, A becomes constant as soon as each component of $\widetilde{C}$ acquires a rational point. As it is enough to prove that $\operatorname{Br}(C)[q] = 0$ for each prime q, we may replace k with its maximal prime-to-q extension, which is pseudo-algebraically closed; equivalently, every absolutely irreducible variety over k admits a k-rational point. In particular, A is isomorphic to the constant sheaf $\mathbf{Z}^r$, and every G-torsor and $\mathbf{Pic}^0_{\widetilde{C}/k}$-torsor is trivial. Together with the fact that $\mathrm{H}^1(k, \mathbf{Z}^r) = 0$, it follows by considering the above exact sequences that $\mathrm{H}^1(k, \mathbf{Pic}_{C/k}) = 0$. It remains to prove that $\mathrm{H}^0(k, \mathrm{R}^2\, \pi_* \mathbf{G}_{m,C}) = 0$. But, the stalk of $\mathrm{R}^2\, \pi_* \mathbf{G}_{m,C}$ is isomorphic to $\mathrm{H}^2(C_{k^s}, \mathbf{G}_m)$, where k^s is the separable closure of k. Since k^s is algebraically closed (as k is perfect), this group vanishes by [24, Corollaire 1.2]. □

Remark 4.2. There is also a proof that uses Tsen's theorem (resp. class field theory) to treat the regular case and then deduces the general case by deformation from points and a Moret-Bailly type formal gluing argument, but we omit the details here.

Remark 4.3. The conclusion that $\mathrm{Br}(C)[n] = 0$ for n prime-to-p cannot be improved to $\mathrm{Br}(C) = 0$ without assuming that k is algebraically closed. If k is separably closed but not algebraically closed, then $\mathrm{Br}(k[x])$ is nonzero. This example appears already in Auslander and Goldman [5, Theorem 7.5]. Consider the Artin–Schreier extension L of $k(x)$ defined by $y^p - y - x = 0$. The ring $k[x,y]/(y^p - y - x)$ is easily seen to be smooth over k, and hence it is the integral closure of $k[x]$ in L. Since k is not separably closed, there is an element $w \in k$ such that $w \notin k^p$. The algebra

$$k[x]\langle y,z\rangle/(y^p - y - x, z^p - w, zy - yz - z)$$

defines an Azumaya algebra over $k[x]$. For more details, see Gille and Szamuely [20, Section 2.5]. This also explains why the full Brauer group is not $\mathbf{A}^1$-homotopy invariant.

The following lemma shows that the only obstruction to extending an $\mathscr{X}$-twisted locally free sheaf on a curve C inside a surface X is whether or not the determinant extends. It is a direct generalization to the twisted setting of [16, Lemma 5.2], although the proof is slightly different owing to the fact that if the $\boldsymbol{\mu}_n$-gerbe $\mathscr{X}$ is nontrivial, then we cannot make use of an $\mathscr{X}$-twisted line bundle on X.

Lemma 4.4. *Let C be a proper curve in a regular quasi-projective 2-dimensional scheme X over a field k of characteristic $p \geq 0$, and fix a $\boldsymbol{\mu}_n$-gerbe $\mathscr{X} \to X$, where n is prime to p. Suppose that $\mathscr{X}$ has index n and that the Brauer class of $\mathscr{X}$ vanishes on every proper curve in X, e.g., k is separably closed or semi-finite by Lemma 4.1.*

If V is a locally free $\mathscr{X}$-twisted sheaf on C of rank n with $\det V = L|_C$, where $L \in \mathrm{Pic}(X)$, then, possibly after taking a finite prime-to-n extension of k, there exists a locally free $\mathscr{X}$-twisted sheaf W on X such that $W|_C \cong V$ and $\det(W) \cong L$.

Proof. Since $\mathscr{X}$ has index n, there is a locally free $\mathscr{X}$-twisted sheaf E of rank n, see [36, Proposition 3.1.2.1(iii)]. Choose an ample line bundle $\mathscr{O}_X(1)$ on X. To prove the Lemma, we will use the following.

Claim 4.5. There is an integer m, a proper curve $D \subset X$ with $\dim(D \cap C) = 0$ in the linear system $|L(mn) \otimes \det(E^\vee)|$, and an invertible $\mathscr{X}$-twisted sheaf M on D such that there is an exact sequence of $\mathscr{X}_C$-twisted sheaves

$$0 \to E(-m)|_C \to V \to M|_{C\cap D} \to 0.$$

We prove the lemma first assuming the claim. Let

$$\gamma \in \mathrm{Ext}^1_C(M|_{C\cap D}, E(-m)|_C)$$

be the corresponding (nonzero) extension class. Our goal is to lift γ to $\mathrm{Ext}^1_X(M, E(-m))$. Given any open subset $U \subseteq X$ containing C, there is an exact sequence

$$\mathrm{Ext}^1_U(M|_{U\cap D}, E(-m)|_U) \to \mathrm{Ext}^1_C(M|_{C\cap D}, E(-m)|_C) \to \mathrm{Ext}^2_U(M|_U, E(-m)(-C)|_U).$$

Now,

$$\mathscr{E}\!xt^0_U(M|_{U\cap D}, E(-m)(-C)|_U) = 0$$

since $M|_{U\cap D}$ is a torsion sheaf, while

$$\mathscr{E}\!xt^2_U(M|_{U\cap D}, E(-m)(-C)|_U) = 0$$

because $M|_{U\cap D}$ is a locally free sheaf on a curve in U and hence has cohomological dimension 1. From the local-to-global ext spectral sequence, it follows that

$$\mathrm{Ext}^2_U(M|_{U\cap D}, E(-m)(-C)|_U) \cong \mathrm{H}^1(U, \mathscr{E}\!xt^1_U(M|_{U\cap D}, E(-m)(-C)|_U)).$$

If we further choose U to be such that $U \cap D$ is affine and $\dim(X \smallsetminus U) = 0$, then this latter group vanishes since $\mathscr{E}\!xt^1_U(M|_{U\cap D}, E(-m)(-C)|_U)$ is supported on D and so

$$\mathrm{H}^1(U, \mathscr{E}\!xt^1_U(M|_{U\cap D}, E(-m)(-C)|_U)) \cong \mathrm{H}^1(U\cap D, \mathscr{E}\!xt^1_U(M|_{U\cap D}, E(-m)(-C)|_U)|_D) = 0,$$

by Serre's vanishing theorem for the cohomology of a quasi-coherent sheaf on an affine variety. It follows that γ lifts to an extension

$$0 \to E(-m)|_U \to \widetilde{V} \to M|_{U\cap D} \to 0$$

on U such that $\widetilde{V}|_C \cong V$. Let W be $j_*\widetilde{V}$, where $j : U \to X$ is the inclusion. Then, W is reflexive since $X - U$ has codimension 2. By construction it restricts to V on C, and since S is regular and 2-dimensional, W is locally free.

The determinant of W is

$$\det(E(-m)) \otimes \det(M) \cong \det(E)(-mn) \otimes \det(M).$$

Since M is a locally free $\mathscr{X}$-twisted sheaf of rank 1 on D, $\det(M) \cong \mathscr{O}_X(D)$ (see [36, Proposition A.5]). But D was chosen to be in the class of the linear system associated to $L(mn) \otimes \det(E^\vee)$. It follows immediately that $\det(W) \cong L$, as desired.

Now we prove Claim 4.5. For sufficiently large m, a general map in $\mathrm{Hom}(E|_C, V(m))$ is injective and has a cokernel isomorphic to the pushforward of an invertible $\mathscr{X}$-twisted line bundle on a general member of the linear system $|N|_C|$, where $N|_C = \det(V(m)) \otimes \det(E|_C^\vee)$. (We suppress the fact that N depends on m in the notation.) This follows from [35, Corollary 3.2.4.21]; if k is finite, to use the required Bertini theorem we can take arbitrarily large finite prime-to-n extensions of k to ensure the existence of rational points avoiding the "forbidden cone" (as any open subset of affine space over an infinite field contains rational points). By assumption, the line bundle $N|_C$ is the restriction of the line bundle $N = L(mn) \otimes \det(E^\vee)$ on X. For sufficiently large m, N is ample, and a general member of $|N|$ restricts to a general member of $|N|_C|$. We let D be a general regular member of $|N|$ such that $D \cap C$ is the support of an injective map $E|_C \to V(m)$ with cokernel the pushforward of an invertible $\mathscr{X}$-twisted sheaf on $D \cap C$. By hypothesis, the Brauer class of $\mathscr{X}$ vanishes on D, so there is an invertible $\mathscr{X}$-twisted sheaf M on D. This proves the claim. □

Before getting to the main result, we need to extend a standard result about elementary transformations to the case of a strict normal crossings scheme. The case of a regular scheme is handled in [36, Corollary A.7].

Definition 4.6. Suppose that Z is an algebraic stack and that $i : W \subset Z$ is a closed substack. Furthermore, suppose we are given a quasi-coherent sheaf F on Z and a quotient $q : F|_W \to Q$. The *elementary transform* of F along q is defined to be the kernel of the morphism $F \to i_*Q$ induced by the adjunction map and q.

Lemma 4.7. *Let X be a scheme and $C \subset X$ an effective Cartier divisor with connected component decomposition $C = \sqcup_i C_i$. Suppose that $\pi : \mathscr{X} \to X$ is a $\boldsymbol{\mu}_n$-gerbe and that E is a locally free $\mathscr{X}$-twisted sheaf. If $q : E|_C \to F$ is a surjection to a locally free $\mathscr{X}$-twisted sheaf F supported on C, then the determinant of the elementary transform of E along q is isomorphic to*

$$\det(E) \otimes \mathscr{O}_X(-\sum_i m_i C_i),$$

where m_i is the rank of $F|_{C_i}$.

Proof. In order for the determinant to be well-defined, we need to check that the subsheaf $G := \ker(E \to i_*F)$ is perfect when viewed as a complex of $\mathscr{X}$-twisted $\mathscr{O}_{\mathscr{X}}$-modules with quasi-coherent cohomology sheaves. In fact, G is locally free. To see this, it suffices to work smooth-locally and prove the following: let Z be a scheme, $i : W \hookrightarrow Z$ an effective Cartier divisor, and E a locally free sheaf on Z. Given a locally free sheaf F on W, the kernel of any surjection $E \twoheadrightarrow i_*F$ is locally free. This in turn reduces to the local case, so we may assume that $Z = \operatorname{Spec} A$, that W is cut out by a single regular element $a \in A$, and that E and F are free on Z and W, respectively. Since $\mathscr{O}_W$ has projective dimension 1 over A, so does F. But since E has projective dimension 0, it follows that the kernel of any such surjection must also be projective, hence locally free, as desired.

To prove the Lemma, it is enough to verify that $\det(i_*F) \cong \mathscr{O}_X(\sum_i m_i C_i)$. Assume henceforth that C is connected, and hence that F has constant rank, say m, everywhere on C. Pulling back to the Severi–Brauer scheme $P \to X$ associated to the Azumaya algebra $\pi_* \mathscr{E}nd(E)$ (so that $\mathscr{X}|_P$ has trivial Brauer class) and using the fact that $\mathrm{Pic}(X) \to \mathrm{Pic}(P)$ is injective, we are immediately reduced to the analogous statement for trivial Brauer classes.

Let L be an invertible $\mathscr{X}$-twisted sheaf. The classical theory of determinants tells us that $\det(i_*F \otimes L^\vee) \cong \mathscr{O}(mC)$. But the rank of i_*F is 0, so this also computes $\det(i_*F)$, as desired. □

Proposition 4.8. *Let X be a quasi-projective geometrically connected snc surface over a field k of characteristic $p \geq 0$, and let $\mathscr{X} \to X$ be a $\boldsymbol{\mu}_n$-gerbe, where n is prime to p. Suppose that $\mathscr{X}$ has index n on each irreducible component of X and that the Brauer class of $\mathscr{X}$ vanishes on each closed subscheme of X of dimension at most* 1. *(This later condition holds when k is separably closed or X is proper and k is semi-finite by Lemma 4.1.) Then there exists a locally free $\mathscr{X}$-twisted sheaf of rank n and trivial determinant on X.*

Proof. First, we show that if there exists an $\mathscr{X}$-twisted sheaf F of rank n on X, then F can be chosen to have trivial determinant. Indeed, for $m \gg 0$, we can assume that $\det(F(m)) = \mathscr{O}_Y(D)$, where D is an effective Cartier divisor on X. By choosing an invertible $\mathscr{X}$-twisted sheaf on D (which is possible by the assumption that the Brauer class of $\mathscr{X}$ vanishes on curves), we can find an invertible quotient Q of $F(m)|_D$. By Lemma 4.7, the elementary transform of $F(m)$ along Q has trivial determinant. Thus, we have constructed a locally free $\mathscr{X}$-twisted sheaf of rank n on Y with trivial determinant.

Now we proceed by induction on the number of irreducible components of X. If X is irreducible (hence regular), then the existence of a locally free $\mathscr{X}$-twisted sheaf F of rank n on X follows from the fact that $\mathscr{X}$ has index n and the existence of Azumaya maximal orders over a regular surface. By the above, we can choose F to have trivial determinant.

In general, let $X = X_1 \cup \cdots \cup X_r$ be the decomposition of X into its irreducible components. Assume that there exists a locally free $\mathscr{X}$-twisted sheaf F of rank n on $Y = X_1 \cup \cdots \cup X_{r-1}$. Let $C = Y \cap X_r$. By the above, we can choose F with trivial determinant. Consequently the restriction of F to C has trivial determinant, which coincides with the restriction $\mathscr{O}_{X_r}|_C$ of the trivial line bundle from X_r. Hence by Lemma 4.4, there exists a locally free $\mathscr{X}$-twisted sheaf F_r on X_r such that $F|_C$ is isomorphic to $F_r|_C$. Let E be the fiber product of F and F_r over their restrictions to C (via the chosen isomorphism) in the abelian category of $\mathscr{X}$-twisted sheaves on X. By applying [40, Theorem 2.1] étale-locally, we see that E is locally free of rank n on X, as desired. By the above, we can choose E with trivial determinant.

By induction, we produce the desired locally free $\mathscr{X}$-twisted sheaf on X. □

The following corollary, which in particular asserts that index equals period, may be found in [36, Corollary 4.2.2.4] in the case when X is smooth over a separably closed field.

Corollary 4.9. *Under the hypotheses of Proposition 4.8, the map*

$$\mathrm{H}^1(X, \mathrm{PGL}_n) \to \mathrm{H}^2(X, \boldsymbol{\mu}_n)$$

is surjective.

Proof. Given a $\boldsymbol{\mu}_n$-gerbe $\mathscr{X} \to X$, the proposition produces a locally free $\mathscr{X}$-twisted sheaf V of rank n. The determinant of V differs from $[\mathscr{X}]$ by a class of $\mathrm{Pic}(X)/n\,\mathrm{Pic}(X)$. Performing an elementary transformation along a suitable effective Cartier divisor corrects the determinant, by Lemma 4.7. □

5. Deformation theory of perfect twisted sheaves

5.1. **Generalities.** The material in this section is similar to [35, Section 2.2.3], except our infinitesimal deformations of the ambient scheme are not assumed to be flat over a base. We review the theory in this case; there are no essential differences.

Let $i : X_0 \hookrightarrow X$ be a closed subscheme of a quasi-separated noetherian scheme X defined by a square-zero sheaf of ideals I of $\mathscr{O}_X$. Let $\pi : \mathscr{X} \to X$ be a $\boldsymbol{\mu}_\ell$-gerbe, write $\mathscr{X}_0 = \mathscr{X} \times_X X_0$ and $\pi_0 : \mathscr{X}_0 \to X_0$ for the restriction of π, and write $\iota : \mathscr{X}_0 \to \mathscr{X}$ for the induced closed immersion. We write $\mathrm{D}_{\mathrm{qc}}^{(1)}(\mathscr{X})$ for the derived category of $\mathscr{X}$-twisted sheaves with quasi-coherent cohomology. Let F_0 be a complex of $\mathscr{O}_{\mathscr{X}_0}$-modules in $\mathrm{D}_{\mathrm{qc}}^{(1)}(\mathscr{X}_0)$.

Definition 5.1.1. A *deformation* of F_0 to $\mathscr{X}$ consists of a complex F in $\mathrm{D}^{(1)}_{\mathrm{qc}}(\mathscr{X})$ and a quasi-isomorphism $\mathscr{O}_{\mathscr{X}_0} \otimes^{\mathrm{L}}_{\mathscr{O}_{\mathscr{X}}} F \simeq F_0$.

For convenience, we write $I \otimes^{\mathrm{L}} F$ for the complex of $\mathscr{X}$-twisted sheaves $\pi^* I \otimes^{\mathrm{L}}_{\mathscr{O}_{\mathscr{X}}} F$ and $I \otimes^{\mathrm{L}} F_0$ for the complex of $\mathscr{X}_0$-twisted sheaves $\pi_0^* i^* I \otimes^{\mathrm{L}}_{\mathscr{O}_{\mathscr{X}_0}} F_0$.

Lemma 5.1.2. *If F_0 is perfect and F is a deformation of F_0 to $\mathscr{X}$, then F is perfect.*

Proof. Note that there is a distinguished triangle $I \otimes^{\mathrm{L}} F \to F \to \iota_* F_0$ in $\mathrm{D}^{(1)}_{\mathrm{qc}}(\mathscr{X})$. If we prove that F has finite Tor-amplitude, then it will follow from [46, Theorem 2.5.5] that $I \otimes^{\mathrm{L}} F$ is quasi-isomorphic to

$$\iota_* \left(I \otimes^{\mathrm{L}} \mathrm{L}\iota^* F\right) \simeq \iota_*(I \otimes F_0).$$

On a quasi-separated noetherian scheme Y the perfect complexes of $\mathscr{O}_Y$-modules are precisely those complexes which have coherent cohomology sheaves and which moreover have bounded Tor-amplitude (see [46, Example 2.2.8 and Proposition 2.2.12]). Thus, choosing an étale covering of X splitting $\mathscr{X} \to X$, we see that the same holds for complexes on $\mathscr{X}$. Since $\iota_* F_0$ and $\iota_*(I \otimes F_0)$ have coherent cohomology sheaves, it follows that if we show that F has finite Tor-amplitude, the lemma will follow.

Recall that a complex F of $\mathrm{D}^{(1)}_{\mathrm{qc}}(\mathscr{X})$ has Tor-amplitude contained in an interval with integer endpoints $[a,b]$ if and only if $\mathscr{T}\!or_n^{\mathscr{O}_{\mathscr{X}}}(G,F) = 0$ for all $\mathscr{O}_{\mathscr{X}}$-modules G and all $n \notin [a,b]$. Suppose that F_0 has Tor-amplitude contained in $[a,b]$. Suppose that $n \in \mathbf{Z}$, and suppose that $\mathscr{T}\!or_n^{\mathscr{O}_{\mathscr{X}}}(G,F)$ is not zero. Then, there is a closed point x of X such that $\mathscr{T}\!or_n^{\mathscr{O}_{\mathscr{X}}}(k(x),F)$ is not zero. But

$$F \otimes^{\mathrm{L}}_{\mathscr{O}_{\mathscr{X}}} k(x) \simeq F \otimes^{\mathrm{L}}_{\mathscr{O}_{\mathscr{X}}} (\mathscr{O}_{\mathscr{X}_0} \otimes^{\mathrm{L}}_{\mathscr{O}_{\mathscr{X}_0}} k(x)) \simeq F_0 \otimes^{\mathrm{L}}_{\mathscr{O}_{\mathscr{X}_0}} k(x).$$

Hence,

$$\mathscr{T}\!or_n^{\mathscr{O}_{\mathscr{X}}}(k(x),F) \cong \mathscr{T}\!or_n^{\mathscr{O}_{\mathscr{X}_0}}(k(x),F_0),$$

which implies that $n \in [a,b]$, as desired. ∎

Definition 5.1.3. Recall that there is an essentially unique *determinant* functor

$$\det : \mathrm{Perf}(\mathscr{X}) \to \mathrm{Pic}(\mathscr{X})$$

that associates to each perfect complex of $\mathscr{X}$-twisted sheaves an invertible sheaf. Given a perfect complex $F \in \mathrm{Perf}(\mathscr{X})$ and an fppf cover $Y \to \mathscr{X}$ over which $F \simeq C^\bullet$, a finite complex of locally free sheaves, the determinant is computed as

$$\det(F) = \bigotimes_{n \in \mathbf{Z}} \det(C^n)^{(-1)^n}.$$

(See [32, Theorem 2], [35, Definition 2.2.4.1].)

Definition 5.1.4. Suppose that F has $\det(F_0) \cong \mathscr{O}_{\mathscr{X}_0}$, and fix one such trivialization. An *equideterminantal deformation* of F_0 is a deformation F as above together with a deformation of the trivialization of the determinant.

The next proposition is well-known to experts and follows immediately from the techniques in [27], cf. [35, Proposition 2.2.4.9].

Proposition 5.1.5. *Let $X_0 \hookrightarrow X$ be a closed subscheme of a quasi-separated noetherian scheme X defined by a square-zero sheaf of ideals I of $\mathscr{O}_X$. Fix a $\boldsymbol{\mu}_\ell$-gerbe $\mathscr{X} \to X$, and let $\mathscr{X}_0 = \mathscr{X} \times_X X_0$. Suppose F_0 is a perfect complex of $\mathscr{X}_0$-twisted sheaves. Then the obstruction to the existence of an equideterminantal deformation of F_0 to $\mathscr{X}$ lies in*

$$\mathbf{H}^2(X_0, I \otimes^{\mathrm{L}} s\mathbf{R}\mathscr{E}\!nd(F_0)),$$

where $s\mathbf{R}\mathscr{E}\!nd(F_0)$ denotes the trace zero part of the complex of endomorphism sheaves.

Remark 5.1.6. In Proposition 5.1.5, when the rank of F_0 is invertible on X_0, we can compute $\mathbf{H}^2(X_0, I \otimes^{\mathrm{L}} s\mathbf{R}\mathscr{E}nd(F_0))$ as $\mathrm{Ext}^2_{\mathscr{X}_0}(F_0, I \otimes F_0)_0$, the kernel of the trace map

$$\mathrm{Ext}^2_{\mathscr{X}_0}(F_0, I \otimes F_0) \to \mathrm{H}^2(\mathscr{X}_0, I)$$

on cohomology. Indeed, in this case the trace map $\mathbf{R}\mathscr{E}nd(F_0) \to \mathscr{O}_{\mathscr{X}_0}$ splits because the composition

$$\mathscr{O}_{\mathscr{X}_0} \to \mathbf{R}\mathscr{E}nd(F_0) \to \mathscr{O}_{\mathscr{X}_0}$$

is multiplication by the rank of F_0 (see [35, Lemma 2.2.4.5]).

5.2. **Fracking.** In the next two sections we describe a standard trick in deformation theory that kills obstructions in dimension 2. Because the general local tool we use roughly corresponds to "punching holes" in a sheaf, we call this *fracking*.

Let X be a locally noetherian scheme, $\pi : \mathscr{X} \to X$ a $\mathbf{G}$-gerbe for some closed subgroup $\mathbf{G} \subset \mathbf{G}_m$, and F a locally free $\mathscr{X}$-twisted sheaf of finite rank. Let $i : \mathrm{Spec}\, K \to X$ denote a closed immersion whose image is a regular point x of X, where K is a field. We will write $\mathscr{X}_0 = \mathscr{X} \times_X \mathrm{Spec}\, K$, we will let $\pi_0 : \mathscr{X}_0 \to \mathrm{Spec}\, K$ denote the restriction of π, and we will let $\iota : \mathscr{X}_0 \to \mathscr{X}$ denote the natural closed immersion.

Given two $\mathscr{O}_X$-modules M and N, there is a trace map

$$\mathrm{Hom}(M \otimes F, N \otimes F) \to \mathrm{Hom}(M, N)$$

induced by the isomorphism

$$\mathscr{H}om(M \otimes F, N \otimes F) \cong \mathscr{H}om(M, N) \otimes \mathscr{H}om(F, F)$$

and the usual trace map. We will let $\mathrm{Hom}(M \otimes F, N \otimes F)_0 \subset \mathrm{Hom}(M \otimes F, N \otimes F)$ denote the kernel of this trace map.

Lemma 5.2.1 (Fracking Lemma)**.** *With the above notation, suppose that*

1. $\dim \mathscr{O}_{X,x} \geq 2$*;*
2. *the rank of F is prime to the characteristic of K;*
3. *M and N are invertible in a neighborhood of x;*
4. *the class of $\mathscr{X}_0$ in $\mathrm{Br}(K)$ is trivial;*
5. *f is an element of $\mathrm{Hom}(M \otimes F, N \otimes F)_0$ whose image in $\mathrm{Hom}(\iota^*(M \otimes F), \iota^*(N \otimes F))_0$ is non-zero;*

Then there exists a locally free $\mathscr{X}_0$-twisted sheaf Q of rank $\mathrm{rk}(F) - 1$ and a surjection $\iota^* F \to Q$ such that, writing G for the kernel of the adjoint map $F \to \iota_* Q$, the endomorphism f is not in the image of the natural inclusion

$$\rho : \mathrm{Hom}(M \otimes G, N \otimes G)_0 \to \mathrm{Hom}(M \otimes F, N \otimes F)_0$$

induced by the canonical isomorphism $G^{\vee\vee} \to F$.

Proof. Since $\mathscr{X}_0$ has trivial Brauer class, we can choose an invertible $\mathscr{X}_0$-twisted sheaf Λ; the sheaf Λ is unique up to non-unique isomorphism. Define two functors

$$\tau_\dagger : \mathrm{Coh}^{(1)}(\mathscr{X}_0) \to \mathrm{Coh}(\mathrm{Spec}\, K)$$

and

$$\tau^\dagger : \mathrm{Coh}(\mathrm{Spec}\, K) \to \mathrm{Coh}^{(1)}(\mathscr{X}_0)$$

by the formulas $\tau_\dagger(A) = (\pi_0)_*(A \otimes \Lambda^\vee)$ and $\tau^\dagger(B) = \Lambda \otimes \pi_0^* B$. It follows from these formulas and the basic theory of twisted sheaves that $\tau^\dagger$ and $\tau_\dagger$ are essentially inverse equivalences. Given a quasi-coherent sheaf M on X and an object $A \in \mathrm{Coh}^{(1)}(\mathscr{X}_0)$, there is a natural isomorphism

$$\tau_\dagger(\iota^* \pi^* M \otimes A) \cong i^* M \otimes \tau_\dagger(A). \tag{5.2.1.1}$$

Write $\overline{F} = \tau_\dagger(\iota^* F)$. By equation (5.2.1.1) and the fact that $i^* M$ and $i^* N$ are 1-dimensional K-vector spaces, we can transport $\iota^* f$ to a non-trivial traceless homomorphism

$$\overline{f} : i^* M \otimes \overline{F} \to i^* N \otimes \overline{F}$$

of K-vector spaces of the same (finite) dimension, see [36, Theorem 3.1.1.11]. Because the nonzero $\overline{f}$ has trace zero and all nonzero scalar matrices have nonzero trace (by the assumption that F has rank prime to the characteristic of K), there is a line $\overline{L}$ in $\overline{F}$ such that $\overline{f}$ does not preserve $\overline{L}$ (i.e., $\overline{f}(i^*M \otimes \overline{L})$ is not contained in $i^*N \otimes \overline{L}$).

Let Q be the $\mathscr{X}_0$-twisted sheaf $\pi^\dagger \overline{F}/\pi^\dagger \overline{L}$ and $\sigma : F \to \iota_* Q$ the adjoint of the natural surjection. Write G for the kernel of σ and $\gamma : G \to F$ for the inclusion. Note that the canonical map $G^{\vee\vee} \to F$ is an isomorphism. (Here we use that F is locally free, hence G is locally free away from x, and that x itself is a regular point.) Since M and N are invertible near x, it follows that there is an induced canonical inclusion

$$\rho : \mathrm{Hom}(M \otimes G, N \otimes G)_0 \hookrightarrow \mathrm{Hom}(M \otimes F, N \otimes F)_0.$$

(The one subtle point is the preservation of the trace zero condition. This follows since M and N are invertible near x and F is locally free, so the traceless condition can be detected on the punctured neighborhood of x.)

This canonical inclusion has the property that for any $s \in \mathrm{Hom}(M \otimes G, N \otimes G)$ we have a commuting diagram

$$\begin{array}{ccc} M \otimes G & \xrightarrow{s} & N \otimes G \\ {\scriptstyle M\otimes\gamma}\downarrow & & \downarrow{\scriptstyle N\otimes\gamma} \\ M \otimes F & \xrightarrow[\rho(s)]{} & N \otimes F. \end{array}$$

It follows that the image of ρ lies in the subgroup B of $\mathrm{Hom}(M\otimes F, N\otimes F)_0$ of those trace zero endomorphisms whose restrictions to the fiber over x map the flag $i^*M \otimes \pi^\dagger \overline{L} \subseteq i^*M \otimes \pi^\dagger \overline{F}$ into the flag $i^*N \otimes \pi^\dagger \overline{L} \subseteq i^*N \otimes \pi^\dagger \overline{F}$. On the other hand, there is an exact sequence

$$0 \to B \to \mathrm{Hom}(M \otimes F, N \otimes F)_0 \to \mathrm{Hom}\left(M \otimes \pi^\dagger \overline{L}, N \otimes \left(\pi^\dagger \overline{F}/\pi^\dagger \overline{L}\right)\right).$$

Since the endomorphism f we started with is nonzero on the right, it is not contained in B, and hence is not in the image of ρ, as desired. □

5.3. **Removing global obstructions by fracking.** In this section, we explain how to use Lemma 5.2.1 to produce unobstructed twisted subsheaves in dimension 2.

Situation 5.3.1. Suppose that X is a proper Gorenstein surface over a field k that is either semi-finite or separably closed, $\mathscr{X} \to X$ is a $\boldsymbol{\mu}_\ell$-gerbe, F is a perfect coherent $\mathscr{X}$-twisted sheaf whose rank is invertible in k, and M is a coherent sheaf on X that is the pushforward of an invertible sheaf on a closed subscheme X' of X that contains a nonempty open subscheme $U \subset X$ (for example, M could be an invertible sheaf on a component of X).

In Situation 5.3.1, there are two trace maps

$$\mathrm{Hom}(M \otimes F, \omega_X \otimes F) \to \Gamma(X, M)$$

and

$$\mathrm{Ext}^2_{\mathscr{X}}(F, M \otimes F) \to \Gamma(X, M).$$

Via Serre duality, there is an isomorphism of trace zero subspaces

$$\mathrm{Ext}^2_{\mathscr{X}}(F, M \otimes F)_0 = \mathrm{Hom}(M \otimes F, \omega_X \otimes F)_0^\vee.$$

Proposition 5.3.2. *In Situation 5.3.1, there is an $\mathscr{X}$-twisted subsheaf $G \subset F$ such that*

(1) *F/G is supported at finitely many regular closed points of X whose residue fields are separable extensions of k, and*
(2) $\mathrm{Ext}^2_{\mathscr{X}}(G, M \otimes G)_0 = 0$.

Proof. Let $f : M \otimes F \to \omega_X \otimes F$ be a nonzero element of the k-vector space of trace zero homomorphisms $\mathrm{Hom}(M \otimes F, \omega_X \otimes F)_0$. Choose a regular closed point x of X with separable residue field such that f is nonzero at x. This is possible since M is invertible on X' and F is locally free on a dense open of X'. Since $M \otimes F$ and $\omega_X \otimes F$ are isomorphic at x, we can apply Lemma 5.2.1 at x (using the fact that $\kappa(x)$ has trivial Brauer group and the fact that the fibers of ω_X and M are one-dimensional) to obtain a subsheaf $G \subseteq F$ such that

(1) G is a perfect sheaf with reflexive hull F,
(2) the map

$$\mathrm{Hom}(M \otimes G, \omega_X \otimes G)_0 \to \mathrm{Hom}(M \otimes F, \omega_X \otimes F)_0$$

induced by passing to reflexive hulls are injective, but
(3) f is not in the image.

Since (2) and (3) imply that that the dimension of $\mathrm{Hom}(M \otimes G, \omega_X \otimes G)_0$ is strictly smaller than that of $\mathrm{Hom}(M_i \otimes F, \omega_X \otimes F)$, we can, possibly after repeating the process finitely many times, find G such that

$$\mathrm{Ext}^2_{\mathscr{X}}(G, M \otimes G)_0^\vee = 0,$$

as desired. □

6. Proofs of the main results

In this section we provide the proofs of Theorems 1.1 and 1.4. As a standard reduction, we may, first of all, assume that $\mathrm{per}(\alpha) = \ell$ is a prime distinct from the residue characteristics of X, see [3, Proof of Theorem 6.2]. Since the index is preserved under taking prime-to-ℓ extensions, we may adjoin a primitive ℓth root, if necessary, so that we can apply the results of Section 2.2 to classes in the Brauer group. By Proposition 3.6 and the results of Section 2.5 (specifically Example 2.5.2 for $\ell > 3$ and Example 2.5.3 for $\ell \mid 6$), the proofs of Theorem 1.1 and Theorem 1.4 both reduce to proving Theorem 1.4 under the additional hypothesis that $X \to \mathrm{Spec}\, R$ is quasi-semistable. Let $\mathscr{X} \to X$ be the $\boldsymbol{\mu}_\ell$-gerbe associated to α.

Let π be a uniformizing parameter of R, so that (π) denotes the sheaf of ideals in $\mathscr{O}_X$ that cuts out the special fiber X_0, and let $I \supseteq (\pi)$ be the sheaf of ideals in $\mathscr{O}_X$ that cuts out the *reduced* special fiber $X_{0,\mathrm{red}} \subseteq X_0$. Write $\mathscr{X}_{0,\mathrm{red}} \to X_{0,\mathrm{red}}$ for the restriction $\mathscr{X} \times_X X_{0,\mathrm{red}}$. By Proposition 4.8, there exists an $\mathscr{X}_{0,\mathrm{red}}$-twisted sheaf F of rank ℓ with trivial determinant. To finish the proof, it suffices to find a perfect twisted subsheaf $G \subset F$ such that $\mathrm{rank}(G) = \mathrm{rank}(F)$ such that G deforms to an $\mathscr{X}$-twisted sheaf over the formal scheme $\widehat{X}$. Indeed, by the Grothendieck Existence Theorem [22, Théorème 5.1.4], any such formal deformation algebraizes to yield an $\mathscr{X}_{\widehat{R}}$-twisted sheaf of rank ℓ on $X_{\widehat{R}}$, the pullback of $X \to \mathrm{Spec}\, R$ to the completion $\widehat{R}$ of R. By Artin approximation, there is thus a coherent $\mathscr{X}$-twisted sheaf V of rank ℓ. By [36, Proposition 3.1.2.1], we have that $\mathrm{ind}(\alpha_{\kappa(X)})$ divides ℓ, as desired.

The rest of this section is devoted to producing the desired formal deformation. We will do this by analyzing the formal local structure of X near $X_{0,\mathrm{red}}$ and then applying Lemma 5.2.1 to eliminate obstructions to deforming across infinitesimal neighborhoods of $X_{0,\mathrm{red}}$.

Given two sheaves of ideals I_1 and I_2 on a scheme Y, define

$$I_1 \diamond I_2 = (I_1 I_2 : I_1 \cap I_2).$$

If $f \in \Gamma(Y, \mathscr{O}_Y)$ is an everywhere regular section, then we have $(fI_1 : fI_2) = f(I_1 : I_2)$ and thus $fI_1 \diamond fI_2 = f(I_2 \diamond I_2)$. If Y is the spectrum of a UFD, then for any two sections $f, g \in \Gamma(Y, \mathscr{O}_Y)$, we have

$$(f) \diamond (g) = (\gcd(f, g)) \tag{6.0.0.1}$$

Since this can be checked locally, it also follows that (6.0.0.1) holds in any locally factorial scheme.

Since $I/(\pi)$ is nilpotent, there is a least m such that $I^m \subseteq (\pi)$. Given $1 \le a \le m$ and $b \ge 0$, let $J_{a,b} = I^a(\pi^b) \diamond (\pi^{b+1})$. The ideals $J_{a,b}$ have the following properties.

(1) $J_{a+1,b} \subseteq J_{a,b}$ for $1 \le a \le m-1$ and $J_{1,b+1} \subseteq J_{m,b}$.
(2) $J_{m,b} = (\pi^{b+1})$. Indeed, $I^m(\pi^b) \subseteq (\pi^{b+1})$, so that $(\pi^{b+1}) \subseteq I^m(\pi^b) \diamond (\pi^{b+1})$. Since X is regular, I is locally principal, so that the inclusion $(\pi^{b+1}) \subseteq J_{m,b}$ is locally an equality and hence an equality.
(3) $J_{a,b}/J_{a+1,b} \cong J_{a,0}/J_{a+1,0}$ for $1 \le a \le (m-1)$ and $J_{m,b}/J_{1,b+1} \cong J_{m,0}/J_{1,1} \cong \mathscr{O}_X/I$. This also follows from the fact that π is a regular section of $\mathscr{O}_X$.

Consider the filtration

$$I = J_{1,0} \supset J_{2,0} \supset \ldots \supset J_{m-1,0} \supset (\pi) = J_{m,0} \supset \quad (6.0.0.2)$$
$$J_{1,1} \supset J_{2,1} \supset \ldots \supset J_{m-1,1} \supset (\pi^2) = J_{m,1} \supset$$
$$J_{1,2} \supset J_{2,2} \supset \ldots \supset J_{m-1,2} \supset (\pi^3) = J_{m,2} \supset$$
$$\ldots .$$

By the above calculations, there are only finitely many $\mathscr{O}_X$-modules appearing in the list of successive quotients in this filtration. By our choice of m, all of the successive quotients are nonzero. Moreover, multiplication by I kills any of these $\mathscr{O}_X$-modules, so we can view them as $\mathscr{O}_{X_{0,\mathrm{red}}}$-modules.

Claim. *Each successive quotient in the filtration defined in* (6.0.0.2) *is locally free of rank* 1 *on its support, which consists of the union of a set of components of* $X_{0,\mathrm{red}}$.

The claim is immediate for $J_{m,b}/J_{1,b+1} \cong \mathscr{O}_X/I = \mathscr{O}_{X_{0,\mathrm{red}}}$. For $1 \le a < m$, we verify the claim étale locally, where we can appeal to the étale local structure (3.0.1) of X. Thus, we may assume that our regular scheme is $X = \operatorname{Spec} R[t_1,\ldots,t_n]/(t_1^{a_1}\cdots t_r^{a_r} - \pi)$. In this case, $I = (t_1 \cdots t_r)$ and $(\pi) = (t_1^{a_1}\cdots t_r^{a_r})$. Using (6.0.0.1), we find that

$$J_{a,0} = (t_1^{\min(a,a_1)} \cdots t_r^{\min(a,a_r)}),$$

and hence the quotient $J_{a,0}/J_{a+1,0}$ is isomorphic to

$$\mathscr{O}_X/(t_1^{\epsilon(1,a)} \cdots t_r^{\epsilon(r,a)}),$$

where

$$\epsilon(i,a) = \begin{cases} 0 & \text{if } a_i \le a, \\ 1 & \text{if } a_i > a. \end{cases}$$

Note that, by our choice of m, for any $1 \le a < m$, some $\epsilon(i,a)$ is nonzero. It follows that the successive quotient is (étale locally) isomorphic to the structure sheaf of some collection of components of the reduced special fiber, proving the claim.

For notational simplicity, set $M_0 = \mathscr{O}_{X_{0,\mathrm{red}}}$ and $M_i = J_{i,0}/J_{i+1,0}$ for $1 \le i < m$. We claim that there is a perfect $\mathscr{X}_{0,\mathrm{red}}$-twisted subsheaf $G \subseteq F$ such that F/G is supported in dimension 0 and

$$\operatorname{Ext}^2_{\mathscr{X}_{0,\mathrm{red}}}(G, M_i \otimes G)_0 = 0$$

for $0 \le i < m$. If this is so then the obstruction of Proposition 5.1.5 to deforming any such G (with trivial determinant) through the filtration (6.0.0.2) vanishes, giving the desired formal deformation. But this follows from Proposition 5.3.2 applied in sequence to $M_1, \ldots, M_{m-1}$.

Remark 6.0.1. In the proof, we may have to take a torsion free subsheaf of F in order to remove obstructions to deforming off of $X_{0,\mathrm{red}}$. (We need G to be perfect so that taking its determinant makes sense, and we work with the equideterminantal deformations in order to kill obstruction spaces using Proposition 5.3.2.) In that case, the resulting $\mathscr{X}$-twisted sheaf may not be locally free. Moreover, a reflexive sheaf on a regular threefold need not be locally free, though it will have torsion free fibers over R. Algebraically speaking, this process may yield a maximal order in the division algebra corresponding to α that is not locally free (see [2]).

References

[1] D. Abramovich and K. Karu, *Weak semistable reduction in characteristic 0*, Invent. Math. **139** (2000), no. 2, 241–273. ↑2.4

[2] B. Antieau and B. Williams, *Unramified division algebras do not always contain Azumaya maximal orders*, Invent. Math. **197** (2014), no. 1, 47–56. ↑6.0.1

[3] M. Artin, *Brauer-Severi varieties*, Brauer groups in ring theory and algebraic geometry (Wilrijk, 1981), 1982, pp. 194–210. ↑1, 6

[4] A. Auel, E. Brussel, S. Garibaldi, and U. Vishne, *Open problems in central simple algebras*, Transform. Groups **16** (2011), no. 1, 219–264. ↑1

[5] M. Auslander and O. Goldman, *The Brauer group of a commutative ring*, Trans. Amer. Math. Soc. **97** (1960), 367–409. ↑4.3

[6] S. Bosch, W. Lütkebohmert, and M. Raynaud, *Néron models*, Springer Berlin Heidelberg, 1990. ↑4

[7] N. Bourbaki, *Éléments de mathématique*, Masson, Paris, 1985. Algèbre commutative. Chapitres 1 à 4. ↑2.4

[8] K. Cesnavicius, *Purity for the Brauer group* (2017), available at 1711.06456. ↑2.1.4

[9] J.-L. Colliot-Thélène, *Birational invariants, purity and the Gersten conjecture*, K-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992), 1995, pp. 1–64. ↑2.1.2

[10] ———, *Algèbres simples centrales sur les corps de fonctions de deux variables (d'après A. J. de Jong)*, Astérisque **307** (2006), Exp. No. 949, ix, 379–413. Séminaire Bourbaki. Vol. 2004/2005. ↑1

[11] ———, *Groupe de Chow des zéro-cycles sur les variétés p-adiques (d'après S. Saito, K. Sato et al.)*, Astérisque **339** (2011), Exp. No. 1012, vii, 1–30. Séminaire Bourbaki. Vol. 2009/2010. ↑3, 3

[12] B. Conrad, *Math 776. From normal crossings to strict normal crossings.* http://math.stanford.edu/~conrad/252Page/handouts/crossings.pdf, accessed July 4, 2018. ↑3

[13] D. A. Cox, J. B. Little, and H. K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011. ↑2.3

[14] A. J. de Jong, *Smoothness, semi-stability and alterations*, Inst. Hautes Études Sci. Publ. Math. **83** (1996), 51–93. ↑3

[15] ———, *A result of Gabber*, 2003. preprint. ↑2

[16] ———, *The period-index problem for the Brauer group of an algebraic surface*, Duke Math. J. **123** (2004), no. 1, 71–94. ↑1, 4

[17] D. Eisenbud, *Commutative algebra, with a view toward algebraic geometry*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995. ↑2.4

[18] G. Faltings and C.-L. Chai, *Degeneration of abelian varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 22, Springer-Verlag, Berlin, 1990. With an appendix by David Mumford. ↑2.3

[19] O. Gabber, *Some theorems on Azumaya algebras*, The Brauer group (Sem., Les Plans-sur-Bex, 1980), 1981, pp. 129–209. ↑2.1.4

[20] P. Gille and T. Szamuely, *Central simple algebras and Galois cohomology*, Cambridge Studies in Advanced Mathematics, vol. 101, Cambridge University Press, Cambridge, 2006. ↑1, 4.3

[21] A. Grothendieck, *Éléments de géométrie algébrique. II. étude globale élémentaire de quelques classes de morphismes*, Inst. Hautes Études Sci. Publ. Math. **8** (1961), 222. ↑2.4

[22] ———, *Éléments de géométrie algébrique. III. étude cohomologique des faisceaux cohérents. I*, Inst. Hautes Études Sci. Publ. Math. **11** (1961), 167. ↑6

[23] ———, *Le groupe de Brauer. II. Théorie cohomologique*, Dix Exposés sur la Cohomologie des Schémas, 1968, pp. 67–87. ↑2

[24] ———, *Le groupe de Brauer. III. Exemples et compléments*, Dix Exposés sur la Cohomologie des Schémas, 1968, pp. 88–188. ↑4

[25] D. Harbater, J. Hartmann, and D. Krashen, *Applications of patching to quadratic forms and central simple algebras*, Invent. Math. **178** (2009), no. 2, 231–263. ↑1

[26] R. T. Hoobler, *A cohomological interpretation of Brauer groups of rings*, Pacific J. Math. **86** (1980), no. 1, 89–92. ↑2

[27] L. Illusie, *Complexe cotangent et déformations. I*, Lecture Notes in Mathematics, vol. 239, Springer-Verlag, Berlin-New York, 1971. ↑5.1

[28] ———, *On Gabber's refined uniformization*, 2009. Lectures at the University of Tokyo. ↑3, 3, 3

[29] L. Illusie, Y. Laszlo, and F. Orgogozo, *Travaux de Gabber sur l'uniformisation locale et la cohomologie étale des schémas quasi-excellents*, Séminaire à l'École polytechnique 2006–2008, Astérisque (2014), 363–364. ↑3, 3

[30] K. Kato, *A Hasse principle for two-dimensional global fields*, J. Reine Angew. Math. **366** (1986), 142–183. With an appendix by Jean-Louis Colliot-Thélène. ↑2.2

[31] G. Kempf, F. F. Knudsen, D. Mumford, and B. Saint-Donat, *Toroidal embeddings. I*, Lecture Notes in Mathematics, Vol. 339, Springer-Verlag, Berlin-New York, 1973. ↑2.3, 2.3.5

[32] F. F. Knudsen and D. Mumford, *The projectivity of the moduli space of stable curves. I. Preliminaries on "det" and "Div"*, Math. Scand. **39** (1976), no. 1, 19–55. ↑5.1.3

[33] D. Kozlov, *Combinatorial algebraic topology*, Algorithms and Computation in Mathematics, vol. 21, Springer, Berlin, 2008. ↑2.3

[34] D. B. Leep, *The u-invariant of p-adic function fields*, J. Reine Angew. Math. **679** (2013), 65–73. ↑1

[35] M. Lieblich, *Moduli of twisted sheaves*, Duke Math. J. **138** (2007), no. 1, 23–118. ↑1, 4, 5.1, 5.1.3, 5.1, 5.1.6

[36] ———, *Twisted sheaves and the period-index problem*, Compos. Math. **144** (2008), no. 1, 1–31. ↑1, 1, 1, 4, 4, 4, 5.2, 6

[37] ———, *Period and index in the Brauer group of an arithmetic surface*, J. Reine Angew. Math. **659** (2011), 1–41. With an appendix by Daniel Krashen. ↑1

[38] ———, *The period-index problem for fields of transcendence degree 2*, Ann. of Math. **182** (2015), no. 2, 391–427. ↑1, 1, 1

[39] E. Matzri, *Symbol length in the Brauer group of a field*, Trans. Amer. Math. Soc. **368** (2016), no. 1, 413–427. ↑1

[40] J. Milnor, *Introduction to algebraic K-theory*, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1971. Annals of Mathematics Studies, No. 72. ↑4

[41] R. Parimala and V. Suresh, *Period-index and u-invariant questions for function fields over complete discretely valued fields*, Invent. Math. **197** (2014), no. 1, 215–235. ↑1

[42] S. Payne, *Boundary complexes and weight filtrations*, Mich. Math. J **62** (2013), no. 2, 293–322. ↑2.3.8

[43] A. Pirutka, *A bound to kill the ramification over function fields*, J. Algebra **377** (2013), 173–178. ↑1, 1, 2, 2.2, 2.3, 2.5.1, 2.5.2, 2.5.3

[44] D. J. Saltman, *Division algebras over p-adic curves*, J. Ramanujan Math. Soc. **12** (1997), 25–47. Correction, J. Ramanujan Math. Soc. **13** (1998), no. 2, 125–129. ↑1, 1, 1, 2, 2.2, 2.4

[45] ———, *Cyclic algebras over p-adic curves*, J. Algebra **314** (2007), no. 2, 817–843. ↑2.5.4

[46] R. W. Thomason and T. Trobaugh, *Higher algebraic K-theory of schemes and of derived categories*, The Grothendieck Festschrift, Vol. III, 1990, pp. 247–435. ↑5.1