A Generalization of the Alexandrov & Path Topologies of Spacetime via Linear Orders

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Abstract

Let X be a topological space equipped with a collection $\mathcal{P}$ of continuous paths. Associated to the pair $(X, \mathcal{P})$ we define a path topology that generalizes the $\mathscr{P}$-topology of Hawking et al. [1] Moreover, we axiomatize a collection $\mathcal{P}_{\ll}$ that generates an order that we call an ***NRT*** *full order. This in turn generalizes the chronology order of spacetime defined via timelike paths. We also investigate the associated Alexandrov topology using ideas of Penrose and Kronheimer[2]. We conclude by defining the notion of a strongly path ordered space $(X, \tau, P_{\ll})$ that generalizes the concept of strongly causal spacetimes and prove an equivalence involving these topologies.*

I. Introduction

Spacetimes have been studied as manifolds equipped with a topology of a 4 dimensional space. However, there are some problems with this approach. E.C. Zeeman pointed out in his 1966 paper that considering a manifold with such properties has its drawbacks, listed below:

1. The manifold topology is highly unphysical. Since the latter is considered to have the topology of a a 4 dimensional Euclidean space, one cannot talk about open neighborhoods, without referring to four spheres, the latter of which have no physical meaning in space time.
2. Four dimensional Euclidean spaces have a topology incompatible with space time. The former is homogeneous at every point ('event') however, this is physically not true, as at every event in space time, light cones separate regions of space like separation and time like separation.

He suggested several improvements to this notion of spacetime. Specifically, he suggested a modification to the conventional notion of spacetimes, by instead considering the latter as manifolds with a new topology called the Fine topology, τ_F. His work was later taken by Hawking et al. and reconfigured so that the theory would make more physical sense. One of Hawking and company's main critiques was that while, τ_F was more physically pleasing, the definitions used to define the latter (*straight lines*) has the information about inertial observers a priori so the fact that the linear structure comes out of this theory is not very surprising.

Hawking and company instead opted to construct a similar topology, what they called the path topology τ_P, that they propose is more physical and produces more surprising results with out so much a priori structure.[1] Among one of the flagship differences between τ_P and τ_F is that while the two induce a Euclidean topology on timelike curves, the former forgoes the requirement that such timelike curves be smooth. In fact, τ_P is the finest topology on a manifold M with such quality. More interestingly is the result that shows which particular paths are continuous in this τ_P topology. Such paths are referred to by Hawking as *Feynman paths* which are informally defined as paths on the space (or manifold to be precise) that are continuous (under the inherit topology) but are constrained to move within a light cone. The open sets under this topology are also worth noting, most notably light cones(including the point of reference) are open in this topology (a result that will be of interest later). Most, importantly is the relationship between τ_P and τ_F. Zeeman showed that τ_F was strictly finer that the inherit topology on the space (or manifold), so one would assume that τ_P is also finer and maybe even comparable to τ_F. Hawking showed that while the first assertion was true, the second is not, as there exists open sets in τ_F that when intersected with a continuous path give singleton points, which are not open. [1]

David B. Malament's work with the path topology τ_F of Hawking et al. also deserves much credit. While Hawking and others proved that τ_F had physically useful properties and gave results useful in the study of space time, Malament showed that these sets of of continuous timelike curves gives the topological and differential structure of any space time inherited with τ_F. However, the subtle difference between Malament's and Hawking's study of spacetimes is that Malament relaxes some of the structure that Hawking deemed necessary. Malament hypothesized that requiring a set to be strongly causal was unnecessary. [5] This result is particularly important in the context of this paper, as seeing results while removing structure may have interesting results. However, Malament's as well as this paper shows that some structure is vital, namely the notion of future and past distinguishing is the minimal level of structure that spacetimes (or general spaces) need to make the topology τ_F physically interesting.

The reader may question if the theories developed by the authors mentioned above can be applied to different kinds of spaces (manifolds). Laurents Hudetz's *Linear Structures, Causal Sets and Topology* proposes a method of taking Tim Maudlin's study of linear structures and applying it to the theory of causal sets for the purpose of studying discrete spaces. While this work will focus only on the study of continuous spaces, it is worth noting that Hudetz approach to discrete spaces does produce physically interesting results involving structures used by Hawking and others in the continuous analog. It is worth noting that while Hudetz uses Maudlin's theory to study discrete spaces, she shows that Maudlin's linear structure theory can be easily integrated with the equivalent notions of *Alexendrov-interval topologies* physically known as just the light cone structure. The results of Hudetz work will not be discusses in further detail but it is important to present, at least intuitavely, the connection that the Hudetz work has with the work of the author

- Hudetz's study of causal spaces comes from the axiomatization of Minkowski geometry and order, where the notion of point events is given a structure that differentiates between the notions of *before* and *after*. While this notion may appear simple and intuative, it is, according to Hudetz, all that is needed to show how causal set theory is an appropriate methodology for studying discrete spaces.[4]

The past literature most closely related to the author's present work (as well as the most vital) is the work of R. Penrose and E. Kronheimer on differential topology. The first of such works *Techniques of Differential Topology in Relativity* is the source of one of the main theorems of this paper. While this particular work of Penrose resides closer to the topic of physics, it did provide some insight into what a possible anzats for generalizing spacetimes may look like. The most important result (at least for the purpose of this paper) comes from Penrose's attempt at studying "non-physical" models of space time in order to understand the global structure of the latter. In this work, Penrose presents a specialized form of one of the main theorems presented in this work. The two most notable results include:

- Global Structure: Given a spacetime M (note not a general space), three global properties of the space time are equivalent: [6]
  - M is strongly causal
  - The Alexandrov topology agrees with the [inherit] topology
  - The Alexandrov topology is Hausdorff
- The set of all point in M obeying strong causality is an open set in the inherit

topology.

This work by Penrose is extremely constraining for our purposes as the setting of most if not all the theorems shown is a space time. This extra structure, allows one to use the concepts of geodesics, length of paths, etc. It is his 1966 work alongside E. Kronheimer that serves as the main foundation of this work. In *On The Structure of Causal Spaces*, Penrose and Kronheimer describe a lot of the concepts that are taken for granted in most studies of space time. The work begins with a simple definitions for causal, chronological, and horisomes ordering and imposes this structure on a general topological space. Most notably, the idea of a chronological order being the only order needed to define the other two (as a matter of fact, any one order suffices to define the other two) . This becomes highly important for later works, as Penrose's global strucutre results shown above requires more than the chronological order for validity. The next sections will introduce some of the definition mentioned in this section in more detail. In the third sections we present the results from our studies of generalized spacetimes and their connection to the works presented in this section. The last section will present ideas the author believe may be necessary to completely generalize the properties of space time.

II. Topology Definitions

We begin this section by reviewing definitions from basic topology, which can be found in Munkres.[7]

Definition II.1. A **topology** on a set X is a collection τ of subsets of X having the following properties:

- $\emptyset$ and X are in τ;
- The union of any sub collection of sets in τ is in τ;
- The intersection of any finite subcollection of τ is in τ.

A set X for which a topology τ has been specified will be called a **topological space**. Subsets $\mathcal{U} \in \tau$ are said to be **open**. Examples of topologies are defined below:

Definition II.2. Let X be a non-empty set. We define the **trivial topology** on this set as the collection consisting of only X and $\emptyset$.

Definition II.3. Let X be a non empty set. We define the **discrete topology** on this set as the collection of all subsets of X, i.e. every single subset of X is open under this topology.

A set X can have different topologies defined on it. However, not all topologies on a set X need be **comparable** to one another. By **comparable** we mean that given two topologies on a set X say τ_A and τ_B then either $\tau_A \subseteq \tau_B$ or $\tau_B \subseteq \tau_A$. If neither is the case then we say the two topologies on the set X are **non-comparable**. If, on the other hand, two topologies on a set X are comparable, then we have technical terms to describe their relationship:

Definition II.4. We say that a topology on a space X, τ_A is **finer** than τ_B if $\tau_B \subseteq \tau_A$. Similarly we say that a topology τ_A is **coarser** than τ_B if $\tau_A \subseteq \tau_B$. If instead, the case is that $\tau_A \subsetneq \tau_B$, we say that τ_B is **strictly finer** than τ_A. The **strictly coarser** case is analogous.

Having presented the definition of a topology τ and the notion of open sets, we present another feature of topological spaces known as the Hausdorff condition:

Definition II.5. A topology τ on a space X is **Hausdorff**, if given any two elements $x, y \in X$, there exists two open sets $A, B \in \tau$ such that $x \in A$ and $y \in B$ with $A \bigcap B = \emptyset$

Saying that a space X is equipped with the Hausdorff condition essentially says that if we have two elements in X, we can always find open sets about these two elements that are completely separated (i.e. disjoint). This leads directly into our next definitions and other important qualities that we wish our space to possess.

Definition II.6. We say that a space X is **metrizable** is there exists a metric function d (distance function) on the set X that induces the topology of X.

Definition II.7. Let X be a set with a given topology τ. We say that this topological space is **connected**, if the only two subsets of X that are both open and closed are $\emptyset$ and X.

Studying a metrizable space is more concrete due to its relation to Euclidean space, as we can now speak of distances between points. The connectedness property, in essence, states that there does not exists a way to 'partition' the space X into two parts such that the union of these two parts is the entire space X and their intersection is empty. There does exists a stronger condition on the space X than connectedness, however, and that condition is analogous to the notion that any two points can be connected via some *path*. Given points x and y in X, a path in X from x to y is a continuous map $\gamma : [a,b] \to X$ such that $\gamma(a) = x$ and $\gamma(b) = y$. If every two points in a space X can be joined by a path then X is said to be **path connected**. Equipped with this definition, we can study spaces, that are path connected locally:

Definition II.8. A topological space X is **locally path connected** at a point $x \in X$, if given an open set $U \subseteq X$ with $x \in U$ there exists an open subset V of U that is path connected with $x \in V \subset U$.

III. Properties and Results

In this section we shall be discussing different topologies on the the set X when we refer to our set X with the given topology, will simply refer to this as τ_X. Recall our definition of a path given in the previous section. Consider a collection $\mathcal{P}$ of continuous paths. The next topology generalizes the path topology of Hawking et al:

Definition III.1. Let X be a topological space equipped with a collection $\mathcal{P}$ of continuous paths. We define the *path* topology $\tau_{\mathcal{P}}$ on the space X as the finest topology such that the all the paths in $\mathcal{P}$ are still continuous (in the standard topology τ_X).

We will call subsets in this topology as the *path open* subsets of X. Since $\tau_{\mathcal{P}}$ is the finest topology such that paths in this collection are still continuous then it follows that $\tau_X \subseteq \tau_p$. In the following lemma we characterize the path open subsets of X

Lemma III.1. Let X be a Hausdorff space. An open subset $\mathcal{O} \in \tau_{\mathcal{P}}$ iff $\forall \gamma \in \mathcal{P}$ there exists an open subset $\mathcal{A} \in \tau_X$ such that $\gamma[I] \cap \mathcal{A} = \gamma[I] \cap \mathcal{O}$.

Proof. $\Rightarrow$ Let $\mathcal{O} \in \tau_{\mathcal{P}}$ and let $\gamma \in \mathcal{P}$ be an arbitrary path $\gamma : [a,b] \to X$. By definition of $\tau_{\mathcal{P}}$ $\gamma^{-1}(\mathcal{O})$ is open in $[a.b]$.Thus it follows that $[a,b] \setminus \gamma^{-1}\mathcal{O}$ is closed in [a,b] which is compact Hausdorff. Thus $[a,b] \setminus \gamma^{-1}(\mathcal{O})$ is compact in $[a,b]$. This implies that $\gamma([a,b] \setminus \gamma^{-1}(\mathcal{O}))$ is compact in X. By the Hausdorff condition of X $\gamma([a,b] \setminus \gamma^{-1}(\mathcal{O}))$ is closed in X and thus in $\gamma([a,b])$. It follows that $\gamma(\gamma^{-1}(\mathcal{O}))$ is open in the subspace topology of $\gamma([a,b])$ which implies that there exists an open set $\mathcal{A} \in \tau_X$ so that $\mathcal{O} \bigcap \gamma([a,b]) = \gamma(\gamma^{-1}(\mathcal{O})) = \mathcal{A} \bigcap \gamma([a,b])$.

$\Leftarrow$ Now suppose $\mathcal{O} \subseteq X$ so that $\forall \gamma \in \mathcal{P}$ there exists $\mathcal{A} \in \tau_X$ then

$$\gamma[I] \cap \mathcal{A} = \gamma[I] \cap \mathcal{O} \tag{1}$$

Then since γ is continuous then the following is true:

$$\gamma^{-1}(\gamma[I] \cap \mathcal{O}) = \gamma^{-1}(\gamma[I] \cap \mathcal{A}) = \gamma^{-1}(\mathcal{A}) \tag{2}$$

By the continuity condition on γ then the last equality is equal to $\gamma^{-1}(\mathcal{O})$ s.t. the set $\mathcal{A}$ is open in the path topology τ_p. □

This is nice first result, however, we would like to know more about the two topologies τ_P and τ_X in particular under what conditions the two topologies agree. A reasonable assumption is that we would want the space X to be metrizable, so that we may be able to speak of "arbitrarily small" neighborhoods or "balls of radius ϵ". We show that with the extra structure of metrizability the two topologies agree.

Theorem III.2. Let the space X be metrizable, locally path connected and let $\mathcal{P}$ be the collection of all continuous paths. Then it follows that $\tau_X = \tau_{\mathcal{P}}$

Proof. We are trying to show that $\tau_p = \tau_X$. The fact that $\tau_X \subseteq \tau_p$ comes from Lemma III.1. Thus we only need to show that $\tau_p \subseteq \tau_X$, i.e. every *path open* subset is also **X-open**.

Assume to the contrary, that there exists a path open set $\mathcal{U}$ that is not X-open. This implies that there exists a "bad point" $p \in \mathcal{U}$ such that no open ball around p is contained in $\mathcal{U}$.

Using the metrizability and local-path connectedness of X, we construct a nested sequence of open subsets $\mathcal{B}_n{}_{n=1}^{\infty}$ of X so that the following hold:

1. $\mathcal{B}_n \subseteq ... \subseteq \mathcal{B}_2 \subseteq \mathcal{B}_1$
2. $\mathcal{B}_n$ is path connected
3. $\mathcal{B}_n \subseteq \mathcal{B}_{1/n}(p)$
4. There exists a "bad" point $p_n \in \mathcal{B}_n$ so that $p_n \notin \mathcal{U}$.

It follows by construction that $\bigcap_{n=1}^{\infty} \mathcal{B}_n = p = \bigcap_{n=1}^{\infty} \mathcal{B}_{1/n}(p)$. Now, picking "bad" points $p_{n+1} \in \mathcal{B}_{n+1} \subseteq \mathcal{B}_n$ we define a path γ_n from p_n to p_{n+1} by

$$\gamma_n : \left[\frac{1}{n+1}, \frac{1}{n}\right] \to X \tag{3}$$

We continue linking such points threading down to the point p via a concatenated path:

$$\gamma[0,1] \to X = \begin{cases} p & x = 0 \\ \gamma_n(x) & x \in \left[\frac{1}{n+1}, \frac{1}{n}\right] \end{cases}$$

The path γ is continuous (via the pasting lemma), thus we have a continuous path through p. By construction no point on γ can intersect $\mathcal{U}$ but this is a contradiction. □

Lemma III.1 and Theorem III.2 are useful in demonstrating that even while changing the topology of our spacetime we can still get two seemingly different topologies to agree, given the right conditions on the space.At this point we move from a "generic" topological space X into a close analog of what Hudetz referred to as a "*poset*".[4] However, we need to modify the definition of a partially ordered set to create our generalization of Hudetz & Maudlin's work. [3][4]

Definition III.2. A **NRT** order $<$ is a binary relation of a set X possesing the following qualities $\forall x, y, z \in X$: $\forall x, y, z \in X$

- It is never true that $x < x$. (**N**on-**R**eflexivity)
- If $x < y$ and $y < z$, then $x < z$. (**T**ransitivity)

There is an additional property that we can require from our order. This quality, is almost equivalent to the density criterion for a space from real analysis. We call this property the fullness criterion, and define it below:

Definition III.3. The order $<$ is **full** if $\forall x \in X$ there exists an element p such that $p < x$ and if $p_1 < x$ and $p_2 < x$, there exists a third point q such that $p_1 < q$, $p_2 < q$ and $q < x$, and this holds for the dual $>$.[2]

A simple example of a full **NRT** order, is the > order on the real line; as for all $x \in \mathbb{R}$ there exists $p \in X$ s.t. $p < x$. Also if $p_1 < x$ and $p_2 < x$, there exists a third point q s.t. $p_1 < q$, $p_2 < q$ and $q < x$.

We will call the set X with its underlying order $<$, and ordered set and denote it by $(X, <)$. With each point x of this space we relate subsets that correspond to points before and after each point in question. Note that the following generalize the chronology future in spacetime. Those sets are defined below:

Definition III.4. Let x be a point in the ordered set $(X, <)$ then the **upset** corresponding to x is defined as:

$$\mathcal{I}^+(x) = \{y \in X : x < y\} \tag{4}$$

The **downsets** are defined analogously:

$$\mathcal{I}^-(x) = \{z \in X : z < x\} \tag{5}$$

The reader should note that the above definitions are related to the chronological past/future in spacetime. It would then seem natural that we should be able to "cover" our entire partially order space, with a combination of up sets and down sets. This is the destination we wish to get to, but first we need a couple of definitions. Such combinations are referred to as the *intervals*.[1] In spacetime this definition is analogous to causal diamonds in special relativity:

[1]For this work we only consider *finite* intervals

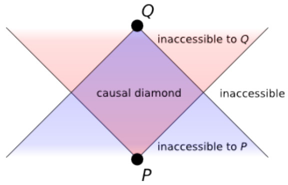

Figure 1: *Casual Diamonds in Spacetime [8]* $\mathcal{I}^+(P)\bigcap\mathcal{I}^-(Q)$

These **intervals** are defined as follows:

Definition III.5. The **intervals** are defined as the sets given by:

$$\mathcal{I}^+(x)\bigcap\mathcal{I}^-(y) = \Big\{z \in X : x < z < y\Big\} \quad (6)$$

Recall that in spacetime, two events p and q are timelike separated iff there is a future-directed smooth timelike curve that goes from p to q.[1] It is also worthwhile investigating certain possible properties of **NRT** order.

Definition III.6. The order $<$ is **future distinguishing** if for any $x, y \in X$ we have that $\mathcal{I}^+(x) = \mathcal{I}^+(y)$ implies that $x = y$. The **past distinguishing** case is similar. [2]

The next definition relates Definition III.5 to a topology on the set X. The latter is known as the *Alexandrov Interval Topology* or *Alexandrov topology* for short. However, there does exists possible sources of confusion, as in the study of spacetime topologies two "Alexandrov" topologies have been used (sometimes) interchangeably. We make the distinction clear here:

Definition III.7. Let $(X, <)$ be an NRT ordered space. We define the **Alexandrov Interval topology** (denoted by τ_{AI}) as the coarsest topology generated by the intervals of the $<$ order.

Definition III.8. The **Alexandrov topology** (denote by τ_A) as the coarsest topology on X in which each upset $\mathcal{I}^+(x)$ and downset $\mathcal{I}^-(x)$ are open in X. However, the latter need not be open in τ_X.

The curious reader, may question if it is possible for the two topologies τ_A and τ_{AI} to be equal. It turns out there does exists a condition guaranteeing that the two topologies are equal.

Lemma III.3. If the underlying order $<$ on the space X is full, then $\tau_A = \tau_{AI}$.

Proof. First note that by definition we are given the following relation between the two topologies

$$\tau_{AI} \subseteq \tau_A \quad (7)$$

This comes from the fact that since the τ_{AI} is generated by the intervals then $\mathcal{I}^+(x)\bigcap\mathcal{I}^-(y) \in \tau_A$. This is easy to see since $\mathcal{I}^+(x)\bigcap\mathcal{I}^-(y) \subseteq \mathcal{I}^+(x)$. Since the order is full it follows that

$$\mathcal{I}^+(x) = \bigcup_{y>x} \mathcal{I}^+(x)\bigcap\mathcal{I}^-(y) \quad (8)$$

Equation 6 can be intuitively explained by the fact that due to the fullness of the order, are guaranteed an element between any two other elements in the space. □

While the fullness criterion of $<$ may seem like a quality separate from the topological qualities of X. However, this is not the case as the next two lemmas suggest:

Lemma III.4. Let X be a set with an *NRT* order $<$. Such order possesses the fullness criterion iff the intervals form a basis for the Alexandrov topology

Proof. $\Longrightarrow$ Note that fullness implies that $\mathcal{I}^+(x)\bigcap\mathcal{I}^-(y)$ are a basis for τ_A. For the latter to form a bais, certain conditions must be met.

- $\forall x \in X$ there exists at least one basis element $\mathcal{I}^+\bigcap\mathcal{I}^-$ must contain x. Since the order $<$ is full then there exists a point $p \in X$ such that $p < x$. It follows that $x \in \mathcal{I}^+(p)$. Since the order $<$ is dual there must also exists a point $p \in X$ $\forall x \in X$ such that $q > x$. The it is clear that $x \in \mathcal{I}^+(p)\bigcap\mathcal{I}^-(q)$
- Let $\mathcal{B}_1 = \mathcal{I}^+(p)\bigcap\mathcal{I}^-(q)$ and $\mathcal{B}_2 = \mathcal{I}^+(s)\bigcap\mathcal{I}^-(t)$. For the intervals to be a basis then if $x \in \mathcal{B}_1\bigcap\mathcal{B}_2$ there musts exists a third basis element $\mathcal{B}_3 \subseteq$ containing x as well. Note that if $x \in \mathcal{B}_1\bigcap\mathcal{B}_2$ then $p < x < q$ and $s < x < t$. By the fullness criterion we have two elements p and s with $p < x$ and $s < x$ there there must exists a third point w such that $p < w$ and $s < w$ and $w < x$. We also have that $x < t$ and $x < q$. By the dual nature of $<$

there exists a third point z such that $t > z$ and $q > z$ but $z > x$. Then it is clear to see that $x \in \mathcal{I}^+(w) \bigcap \mathcal{I}^-(z) \subseteq \mathcal{B}_1 \bigcap \mathcal{B}_2$

$\Longleftarrow$ Now let the intervals $\mathcal{I}^+(p) \bigcap \mathcal{I}^-(q)$ form a basis for τ_A then we need to show that $<$ is a full **NRT** order. Since the intervals are a basis then $\forall x \in X$ at least one basis element $\mathcal{I}^+(p) \bigcap \mathcal{I}^-(q)$ contains x. This implies that $\forall x \in X$ there exists $p, q \in X$ such that $p < x$ and $x < q$. Also, if $x \in \mathcal{B}_1 \bigcap \mathcal{B}_2$ (where $\mathcal{B}_{1,2}$ are the same as before) then this implies that we have $p < x$ and $s < x$. Since the intervals form a basis there exists a third basis element $\mathcal{B}_3 = \mathcal{I}^+(w) \bigcap \mathcal{I}^-(z) \subseteq \mathcal{B}_1 \bigcap \mathcal{B}_2$ that also contains x then this implies that when we have $p < x$ and $s < x$ we also have a third point w in X such that $p < w$ and $s < w$ but $w < x$. The dual of this follows. □

Moreover, it is interesting to see that there is some connection between the Hausdorff condition and the *future/past distinguishing* properties of the order $<$.

Lemma III.5. If $(X, <)$ is an **NRT** ordered space and Alexandrov topology τ_A is Hausdorff and the order $<$ is full then the latter is *future/past distinguishing*.[2]

The curious reader may wish to know how the Alexandrov topology τ_A relates to the given topology on a space X. As the next example shows, the answer to this can vary.

Example. *Let $X = \mathbb{R}^2 = \mathbb{R}^{1+1}$ i.e. the one dimensional time with an absolute time order such that $(x, t) \in X$. Let $p = (x_1, t_1)$ and (x_2, t_2) be two elements of X. We say that $p \ll q$ iff $t_1 < t_2$. Such order is an **NRT** full order, the given topology $\tau_{\mathbb{R}^2}$ is strictly finer than the Alexandrov topology τ_A but the latter is **NOT** Hausdorff. Figure Figure 2 illustrates that the Alexandrov topology lacks the Hausdorff condition as given any two points p, q we can never find two open sets in the Alexandrov topology containing each, that are disjoint.*

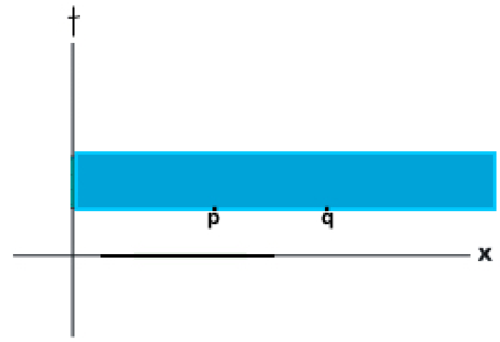

Fig. 2

It is at this point that we are ready to connect what we have discussed thus far about the properties of **NRT** orders and the study of linear order i.e. lines, paths, timelike curves. Recall the collection of paths presented before $\mathcal{P}$. We now show that we can axiomatize such collection using ides from Maudlin's Theory of Linear Structures to generate an **NRT full** order on the space X that generalizes the chronology order on spacetimes. We present the axioms below and refer to the collection of axiomatized paths as $\mathcal{P}_{\ll}$.

Definition III.9. Let $\mathcal{P}_{\ll}$ be the collection of paths in X that satisfy the following axioms:

1. Each path $\gamma : [a, b] \to X$ in $\mathcal{P}$ has a a unique *linear order* on the range of γ such that:
   - If $[c, d] \subseteq [a, b]$ then $\gamma_{|[c,d]} \in \mathcal{P}_{\ll}$.
   - If $h : [c, d] \to [a, b]$ is a continuous increasing function then $\gamma \circ h \in \mathcal{P}_{\ll}$
2. For all $\gamma \in \mathcal{P}_{\ll}$ there does not exists γ such that $\gamma(a) = \gamma(b)$ (no closed loops)
3. Every point in X lies on the *interior* of some path $\mathcal{P}_{\ll}$.
4. For all $x, y, z \in X$ if there exists a path $\gamma_1 \in \mathcal{P}_{\ll}$ with initial point x and endpoint y and another path $\gamma_2 \in \mathcal{P}_{\ll}$ with initial point y and endpoint z, then there exists a third path $\gamma_3 \in \mathcal{P}_{\ll}$ joining x to y. (via the Pasting Lemma).
5. If there is a path $\gamma_1 \in \mathcal{P}_{\ll}$ from p_1 to x and a path $\gamma_2 \in \mathcal{P}_{\ll}$ from p_2 to x then

there is a point $q \in X$ and a path in $\mathcal{P}_{\ll}$ from p_1 to q and p_2 to q and also q to x and the dual of this also holds.

We are now ready to present the following definition:

Definition III.10. Given $p, q \in (X, \mathcal{P}_{\ll})$ we say $p \ll q$ iff there exists a path $\gamma \in \mathcal{P}_{\ll}$ such that $\gamma(a) = p$ and $\gamma(b) = q$. We shall call the order generated by the collection of paths $\mathcal{P}_{\ll}$ the **path order**.

The following lemma shows that the order $\ll$ will attain all of the properties discussed above.

Lemma III.6. Given $(X, \tau_X, \mathcal{P}_{\ll})$ the order $\ll$ is a *full NRT* order.

This result follows immediately, since there does not exists closed loops in this collection of paths due to the unique linear oder on the range of γ. The second condition for a NRT order is satisfied via the fourth axiom. Lastly, the fullness comes from the last axiom. In previous literature, the physical analogous for our path order was the order generated by the smooth timelike curves. The reader may question why our definition of path order lacks the anti-symmetric property; we omitted this property since events in a spacetime can be space like separated and thus we cannot impose this property for our generalized version of timelike ordering.

The main property we wish to generalize is the property of a space Penrose and Malament called *strongly causal*.[6][5] The two author's definition is presented below:

Definition III.11. A spacetime is **strongly causal** iff, for all points p and all open sets $\mathcal{O}$ containing p, there exists an open set $\mathcal{O}_1$ with $p \in \mathcal{O}_1 \subseteq \mathcal{O}$ such that no future directed smooth timelike curve which goes through p and leave $\mathcal{O}_1$ ever returns to $\mathcal{O}_1$.[6][5]

We now present our generalized version of this definition:

Definition III.12. A space X is **strongly path ordered** iff, $\forall p \in X$ and all open sets $\mathcal{O}$ (w.r.t the given topology) containing x, then there exists an open set $\mathcal{O}_1$ with $p \in \mathcal{O}_1 \subseteq \mathcal{O}$ with the property that no path $\gamma \in \mathcal{P}_{\ll}$ which goes through p and leaves $\mathcal{O}_1$ ever returns to $\mathcal{O}_1$.

We are now ready to present the main results of this work, a generalized albeit shortened version of the global structure theorem proved by Penrose in his work on differential topology.

Theorem III.7. Given a path ordered topological space $(X, \tau_X, \mathcal{P}_{\ll})$ so that the upsets/downsets are also in τ_X the following properties are equivalent:

1. The space X is strongly path ordered.
2. The two topologies τ_A & τ_X are equal.

Proof. We first show that 1.) $\Longrightarrow$ 2.). Take the fact that X is strongly path ordered as a given. By definition of the Alexandrov topology $\tau_A \subseteq \tau_X$ we need to prove that with condition 1.) $\tau_X \subseteq \tau_A$. To accomplish this we need only show that given a set $\mathcal{U}$ open in the given topology of X that we can find a set $\mathcal{V}$ open in the Alexandrov topology such that $\mathcal{V} \subseteq \mathcal{U}$. Let $x \in \mathcal{U}$. Since the space X is strongly path ordered given the open set $\mathcal{U}$ there exists an open subset $\mathcal{U}_1$ s.t $x \in \mathcal{U}_1 \subseteq \mathcal{U}$ (with the property that given a path γ the set $\gamma^-1(\mathcal{U}_1) = [c, d] \subseteq [0, 1]$ is connected). This is equivalent to saying that a path γ that passes through $\mathcal{U}_1$ and exits the latter never returns. Then by the fullness property of the space we can see that $\gamma(c) \ll x \ll \gamma(d)$ which implies that $x \in \mathcal{I}^+(\gamma(c)) \bigcap \mathcal{I}^-(\gamma(d))$. However, the latter is a causal diamond which is open in the Alexandrov topology τ_A suggesting that $\mathcal{V} \subseteq \mathcal{U}_1 \subseteq \mathcal{U}$, thus $\tau_X \subseteq \tau_A$.

Now we show that 2.) $\Longrightarrow$ 1.). Since the two topologies are equal, then it follows that $\tau_A \subseteq \tau_X$ and $\tau_X \subseteq \tau_A$. Now let $\mathcal{U}$ be an open set in the given topology of X, then by the fact that $\tau_X \subseteq \tau A$ $\exists$ a causal diamond,$\mathcal{V}$, (an open set in the Alexandrov topology) contained in $\mathcal{U}$, however, using the fact $\tau_A \subseteq \tau_X$ again, $\exists$ another open set (in the inherit topology) call it $\mathcal{U}_1$ s.t. $\mathcal{U}_1 \subseteq \mathcal{V}$. Then we can see that the space X has to be strongly path ordered, as any path that goes through $\mathcal{U}_1$ and leaves the latter must also go through the causal diamond $\mathcal{V}$ and come back, which is not possible. $\square$

IV. Conclusion

We have taken quite a long trip to get to our main theorem. It is worthwhile to recap. We began by introducing the concepts of paths on a topological space. We then took a collection of continuous paths and introduced the path topology, the finest topology on a space X such that our collection of paths is still continuous. We also showed that the only requirements needed to make $\tau_{\mathcal{P}} = \tau_X$ were metrizability and local path connectedness. We then switched our attention to the study of orders namely, the order we called an **NRT** order. Equipping our order with the fullness property allowed us to build the upsets and the downsets, which were critical in being able to study the *Alexandrov* and *Alexandrov interval* topologies. We then connected the properties of orders and paths in the axioms that gave us the order $\ll$. Then, using the generalized version of Maudlin's *strongly causal* we were able to present our main theorem. In future work we will be aiming to add a third property to **Theorem III.7**, namely the Hausdorff condition to attain the full generalized version of Penrose's theorem given in Methods of Differential Topology.

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