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Sandbagging in One-Card Poker

A Thesis in Mathematics

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Abstract

In 1950, Kuhn developed a simplified version of poker involving two players, three cards, and a maximum win/loss of plus and minus 2. Using basic game theory, he was able to show an optimal family of solutions that returns plus and minus $\frac{1}{18}$ to the dealer and opener respectively. In a slightly more complicated version where maximum win/loss becomes plus and minus 3 and includes the possibility of a raise, what is the equilibrium optimal strategy? Importantly, we are interested in whether checkraising is a valid strategy. We find that the expanding the game parameters to include raises does not change the expected value and that an optimal strategy is very similar to that in the game without raises. Additionally, we find that regardless of how many raises are allowed, raises will not be called except with aces and sandbagging does not constitute a strictly dominant strategy.

Acknowledgements

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Sandbagging in One-Card Poker

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Introduction

The game of one card poker is the easiest simplification of the commonly played forms of poker (Texas Hold Em, etc.). This game, played by two people, utilizes a deck of exactly three cards: one queen, one king and one ace with ace high, queen low. After each player antes up, one card is dealt to each player. The third card remains on the sidelines to introduce uncertainty. After the cards are dealt, betting begins and goes up to include a possible raise. In such a simple game, with at most four rounds of betting, with only two players and such a simplified deck, what is the equilibrium strategy?

We believe that the solution to this has been found- in 1995, Koller and Pfeffer published an algorithm that allows them to closely approximate the optimal strategy in two player games[3]. However, it does not appear that the solution to our three card game including raises has been analyzed, nor is the solution available in the literature. A simplified version of our game which excludes the possibility of raises was introduced by Kuhn 1950. He, and authors following him, have found that the equilibrium strategy give one player expected value of 1/18, while his opponent has expected value $-1/18$. We will present an analysis of the strategy for the game involving raises. We know that such an strategy exists both via von Neumann's work on two-player zero-sum games, and via Nash's work on equilibria. This paper will deomstrate and analyze such a set of strategies.

Kuhn Poker

We begin this paper by introducing the simplified version of the game, Kuhn Poker. In 1950, Kuhn published a paper titled "Simplified Two-Person Poker." In Kuhn's version of the game, commonly called "Kuhn Poker," there are 3 cards- 1 queen, 1 king and 1 ace- and two players. Each player antes up 1, and one player gets designated the opener and the other as dealer. The opener begins the game and can either maintain his bet at 1 or increase it to 2. Betting then passes to the dealer who must match the opener's bet, fold, or can increase the bet to 2. Kuhn restricts the maximum amount either player has in the pot to 2− which indicates that exactly one player can actually bet. In general, players in Kuhn Poker can check, call or bet.

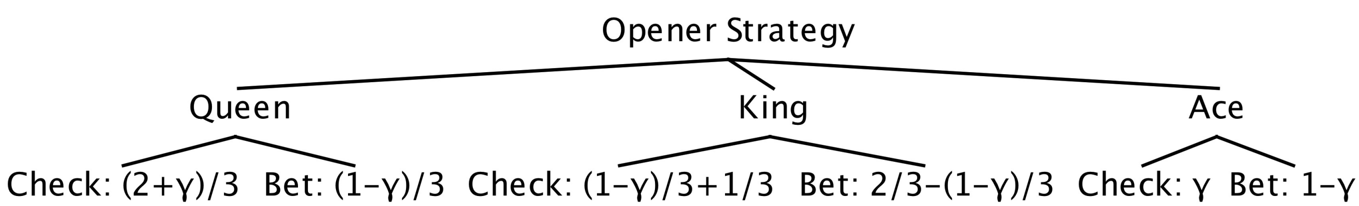
Kuhn takes his game to normal form, and as a result, finds a series of dominant strategies and the expected value of the game. In this case, the expected value for the opener is $-\frac{1}{18}$ and $\frac{1}{18}$ for the dealer. By returning to extensive form, and including the following "behavior parameters:"

$$
\begin{aligned}
&\text{Opener:} \\
&\quad \alpha = \text{probability bet with queen} \\
&\quad \beta = \text{probability check-call with king} \\
&\quad \gamma = \text{probability bet with ace} \\
&\text{Dealer:} \\
&\quad \xi = \text{probability bet with queen} \\
&\quad \eta = \text{probability call with king}
\end{aligned}
$$

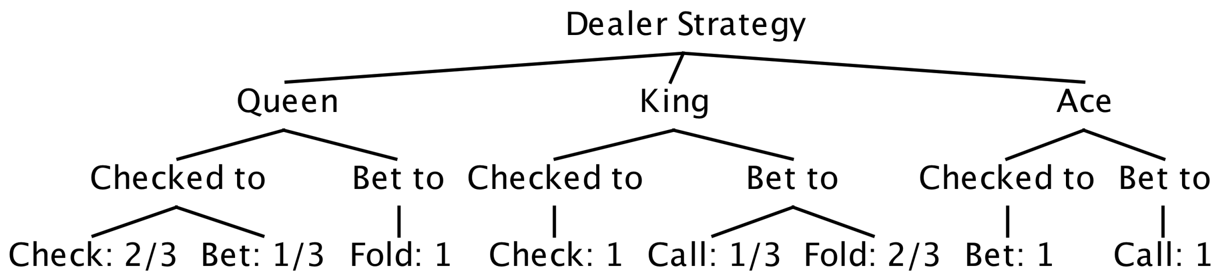
Kuhn describes the family of optimal strategies: [p. 101,[4]]

$$
\begin{aligned}
&\text{Opener:} \\
&\quad \alpha = \frac{\gamma}{3} \\
&\quad \beta = \frac{\gamma}{3} + \frac{1}{3} \\
&\quad \gamma \in [0,1] \\
&\text{Dealer:} \\
&\quad \xi = \frac{1}{3} \\
&\quad \eta = \frac{1}{3}
\end{aligned}
$$

If we alter things slightly such that we change γ to equal the probability the opener checks with the ace, we can create the following tree to describe the opener's strategies. The reason for this variable change will become rapidly apparent in the following sections.



It is clear that the above tree is the same as the one described by Kuhn, with slightly different variable names. We also create the following tree for the dealer's strategy using Kuhn's solution:

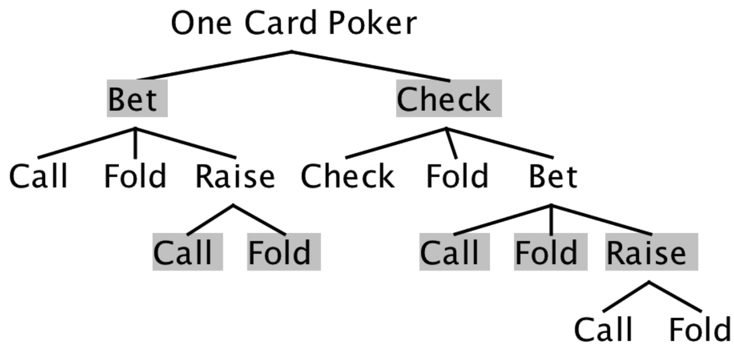


It's interesting to note that in this game, one can choose any probability to bet with the ace and still play an equilibrium strategy. Additionally, even in this simplified version, one notes that bluffing and under-betting are valid strategies. For example, despite the simplicity, one need not bet the ace 100% of the time. Instead, it is a perfectly valid move to lure the opponent into betting more than he should. In other words, much of the strategic complexity in normal poker is preserved in this game.

Revised Kuhn Poker

In this section, we go more into detail about the revised version of Kuhn Poker that includes the possibility of a raise. As we established earlier, the game is played with three cards: 1 queen, 1 king, and 1 ace. In this game, the queen is the low card, king is mid, and ace is high. Furthermore, the game is played between two players, whom we designate the "opener" and the "dealer." The opener will begin the game and thus "open."

Before the game begins, each player puts an ante into the pot. For sake of simplicity, let us say the ante is \$1. Once the pot has been filled, the dealer first deals one card to the opener, and then one card to himself. The remaining card is shelved. Upon looking at his card, the opener now has the choice of checking (keeping the pot at the same amount), or betting \$1. In response, the dealer may check if he is checked to, bet an additional \$1 if he is checked to, fold (giving up the pot), call a bet if he is bet to (matching the \$1), or raise if he is bet to (by another \$1). In response, the opener may raise a bet, call a raise, fold a raise, fold a bet, or call a bet. Finally, the dealer has the option of either calling or folding a raise if he is raised to. The complete game tree is shown below with frivolous folding omitted. The chart below should clarify what moves are available, where the opener's moves are highlighted in grey and the dealer's moves are in white. Moving to a new row in the tree indicates that one more dollar has entered the pot and the opponent must make a decision. At any point, if the game reaches a standoff (for example, a sequence of checks), then the players reveal their cards and the player with the higher card takes the pot. Alternatively, if a player folds, then the other player takes the pot. Thus, a player may win or lose at most \$3.



In the remainder of this paper, we will identify moves by the "complete strategy." That is, whenever either player makes a move, he already knows the move he will make in the future. For example, whenever the opener checks, he may have already decided to raise any bet that comes his way (we would refer to such a strategy as checkraising). The reasoning for this comes from the definition of Nash Equilibrium. In a Nash Equilibrium strategy, the player cannot benefit by unilaterally changing his strategy. This implies that **regardless** of what the opponent does, a player playing by an optimal strategy will maintain at least the same expected value. In rare cases, the opponent may make a deviation that strictly benefits the other player and directly hurts himself. Either way, the player will never be worse off by sticking to his optimal strategy.

Stupid Mistakes

We now begin our analysis of the game in a game theoretical approach by finding dominated strategies. In Swanson 2005, Swanson analyzes Kuhn's poker by identifying several strategies which he deems stupid mistakes- dominated strategies. This paper will reproduce the strategies which were identified as stupid mistakes and analyze them in the context of our new game. We will also introduce a few other "stupid strategies."

1. Folding the ace. This remains a stupid mistake in our game. At equilibrium, where the opponents knows our player's strategy, there is no possibility that folding an ace can lead to a better result. Folding the ace will only result in losing the pool, which the player should have claimed.

2. Calling the queen. This remains a stupid mistake in our game. Calling the queen implies that a bet or raise has just been made and our player has an option to match the bet. Since the queen will lose 100% of the time in a standoff, it makes no sense for the player to match a bet and lose even more money.

3. Dealer checking the ace. This stupid mistake would imply that if the opener checked, then the dealer would check again. To see why this is a stupid mistake, consider his alternative. He could bet the ace, in which case the opener will either match the bet, bluff and raise, or fold. In none of these cases is the dealer strictly better off checking

his ace and accepting the original pot. Indeed, the dealer is strictly worse off in several of these. The dealer checking the ace remains a stupid mistake.

4. Betting the king. Swanson identifies betting the king as a stupid mistake for the following reason. In the original game where a player cannot raise, then betting the king would imply that the opponent would act in exactly one way depending on his card. If the opponent had a queen, then he/she would fold since matching the bet would be a stupid mistake. The original player is no better off in this case. If the opponent had an ace, then the opponent would call the bet. In this case, the original player is strictly worse off. Thus, Swanson identifies betting the king as a stupid mistake. However, in our game, this is no longer true. While the opponent would either call or raise with an ace, the opponent may raise with a queen instead of folding. In such a case, there exists a possible strategy (bet, raise, call) where our player comes out strictly ahead. This implies that betting the king is potentially a valid strategy.

5. Here we introduce the first of our new stupid mistakes. This is raising the king (by either player). The logic here is exactly the same as in Swanson's betting the king. The other player's strategy is strictly dependent upon the card he holds. If the card were an ace, then the raise would be called and our original player strictly worse off. If the card were a queen, then opponent would fold and our original player no better off. Thus, raising the king is a new stupid mistake.

6. The second new stupid mistake is checkcalling the ace by the opener. If the opener has the ace, and the opener checks and the dealer bets, then the opener has a choice between folding the ace (which he will never do), calling the bet or raising the bet. In the case that he calls the bet, the opener takes home \$2. In the case that he raises the bet, the opponent either folds (and the opener wins \$2) or the opponent calls (and the opener wins \$3). We can clearly see checkraising is dominant over checkcalling the ace.

7. The third of our new stupid mistakes is calling the ace by the dealer if he is bet to. As with mistake number six, the dealer is never worse off by raising the ace versus calling it. The opener will either fold or call the raise- either way, the dealer can make at least the \$2 he would have made by calling.

It should be clear from above that there are six stupid mistakes that apply in our game. Unlike Kuhn Poker where we can easily identify all dominated strategies, we are unable to do so in this game. However, we do know that the above strategies are dominated, and going forward, we can assume that neither player will make any of these stupid mistakes. If we make this assumption, then we see that both players have a fairly limited selection of strategies that they can pursue.

The following list demonstrates all choices available to the opener given that he will not make any of the above stupid mistakes:

- If he is dealt a queen: betfold (betting with the intention of later folding), checkfold (checking with the intention of later folding), or checkraise (checking with the intention of later raising). His other two potential strategies (betcall and checkcall) are stupid mistakes. Thus, the probabilities that the opener uses one of these strategies (betfold, checkfold and checkraise) given that he has been dealt a queen should sum to one.

- If he is dealt a king: betcall, betfold, checkcall, or checkfold. His only other potential strategy, checkraise, is a stupid mistake. As above, given that the opener is dealt a king, the probabilities that he utilizes these four strategies will sum to one.
- If he is dealt an ace: betcall or checkraise. It is completely obvious that betfold and checkfold are stupid mistakes. Checkcall is a dominated strategy as shown above. Thus, if the opener is dealt an ace, he will either bet (and call the raise 100% of the time he is given the option) or he will check (and raise 100% of the time he is given the option). Together, the probabilities of these two strategies will sum to one if the opener is dealt the ace.

Similarly, we can analyze the dealer's strategies.

- If he is dealt a queen:
  - if he is checked to: check or betfold. The only other potential strategy is betcalling, which we know is a dominated strategy. Thus, the probability of checking and betfolding will sum to 1 given that he is dealt a queen and is checked against.
  - if he is bet to: fold or raise. Again, the only other potential strategy is calling, which remains dominated. Thus, given that the dealer is dealt a queen and is bet to, the probabilities of folding and raising will sum to 1.
- If he is dealt a king:
  - if he is checked to: check, betcall, or betfold. Every strategy is available to him in this scenario, as none are obviously dominated.
  - if he is bet to: fold or call. The other strategy is raising, and we showed that avenue is dominated. Thus, given that the dealer is dealt the king and bet to, the dealer will either fold or raise.
- If he is dealt an ace:
  - if he is checked to: betcall. We know that both checking the ace and folding the ace are dominated strategies. Thus, if the dealer is checked to and holds the ace, he will betcall.
  - if he is bet to: raise. We know that calling and folding the ace are dominated, so if the dealer is bet to and holds the ace, he will raise. We will see later that calling the ace is not as dominated as we might expect.

At this point, it is worth noting again that these are not necessarily all available to either player in the family of equilibrium strategies. The reason is that there may be strategies written above which end up being dominated. Furthermore, these dominated strategies may lead to other strategies becoming ineffective. For example, one can imagine that if checkraising with the queen ended up being dominated, then raising with the ace would effectively become useless as the opponent would know you hold an ace. This could lead to certain other strategies which appear to be strictly dominated to in turn become only dominated. Indeed, this is the case in the family of solutions described below.

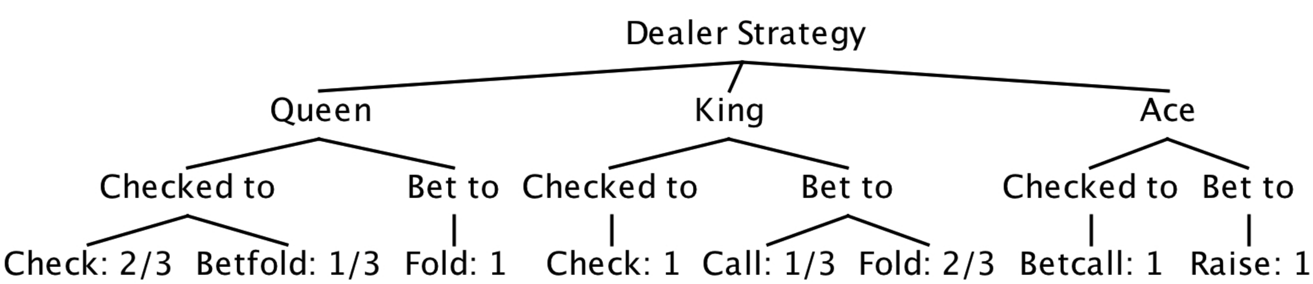
Family of Solutions for Opener and Dealer

When one takes the game into extensive form, one creates a single equation to describe the expected value of the game. It is possible to make the equation bilinear, which is highly convenient for problems of optimization, as taking partial derivatives yield linear equations. Optimizing this equation, contingent upon certain constraints, yields families of solutions. Although additional tricks such as parameterization are required in order to the get the equations into workable forms, we are able to ultimately arrive at a family of solutions.

It should be noted that this family of solutions is not guaranteed to be unique. Indeed, we will show later that other strategies which are not captured in this solution are also optimal. Aside from these families of solutions, it is possible that there exist other optimal solutions which give the same expected value.

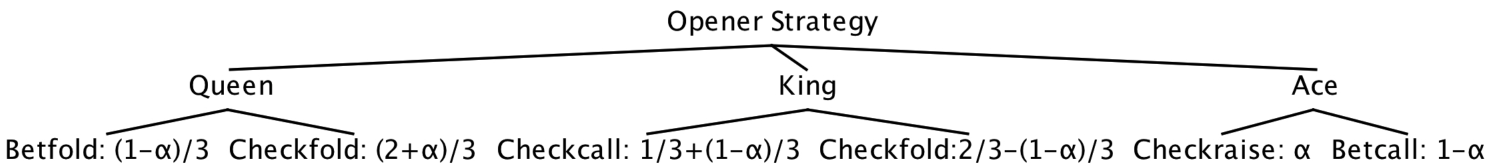
Our particular family of solutions describes a unique strategy for the dealer. He will check the queen with 2/3 probability, and betfold with 1/3 probability. This assumes that he was checked to. If the dealer is bet to and holds the queen, he will fold with 100% certainty. If the dealer is checked to and holds a king, he will check it with 100% certainty. If the dealer is bet to, he will fold with 2/3 probability and call with 1/3 probability. If the dealer is bet to and holds the ace, he will raise with 100% probability. Alternatively, if the dealer is checked to, he will betcall with 100% probability.

The dealer's strategy is represented graphically below:



The solution set for the opener is slightly more complicated. The opener can choose any probability for checkraising with an ace between zero and one. Let us call this probability α. Then the probability of betcalling with an ace is $1-\alpha$. Bluffing with a queen (betfolding) turns out to be $\frac{1-\alpha}{3}$ and checkfolding is $1-\frac{1-\alpha}{3}$. Checkcalling with the king turns out to be $\frac{1}{3}+\frac{1-\alpha}{3}$ and checkfolding with the king is $\frac{2}{3}-\frac{1-\alpha}{3}$.

The opener's strategy is represented graphically below:



We notice that both of these strategies look awfully similar to the strategies presented for Kuhn's poker. We will elaborate more on the similarities later.

Proving Equilibrium for the Opener

We now show that the opener's family of strategies actually belong to the set of optimal strategies. Since the opener always moves first, he always has a choice between checking and betting. In each case where the opener is blessed with choice, we must show that each choice provides equal expected value. Additionally, although our strategy specifies that the opener will perform certain actions with 0 probability, we verify that these strategies are indeed dominated and that the opener is strictly better off by not performing these actions.

Opener Queen

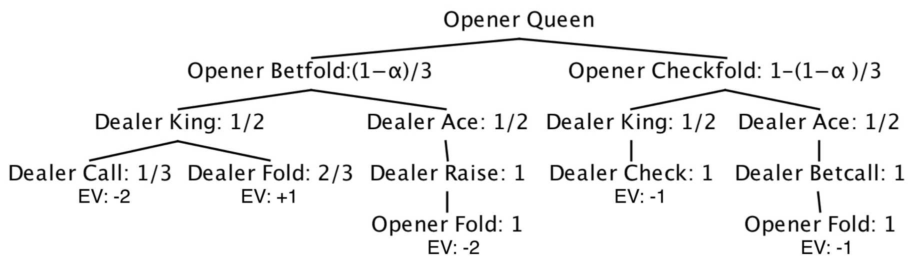
We begin by analyzing what happens when the opener has the queen. According to our strategy, he will bluff the queen (betfold) with $\frac{1-\alpha}{3}$ probability and checkfold with $1-\frac{1-\alpha}{3}$ probability.

Let us first consider bluffing (betfolding). Half the time the dealer will have the king and half the time the dealer will have the ace. If bet to while holding the king, the dealer will call with $\frac{1}{3}$ probability and fold with $\frac{2}{3}$ probability. Therefor, the opener will have: $\frac{1}{2}[(\frac{1}{3}(-2)+\frac{2}{3}(+1)]=0$ if the dealer has the king. The other half the time, the dealer will have the ace. If bet to, the dealer will raise 100% of the time, to which the opener will respond by folding. Thus, $\frac{1}{2}(-2)(1)=-1$ is the expected value from the dealer having the ace. Overall, the return from bluffing the queen is (-1).

It is easy to see that betcalling remains dominated. We see that the dealer with an ace will respond to a bet by raising so calling this raise is strictly worse than folding the raise. Additionally, we see that betcalling and betfolding produce the same result when the dealer has the king. Thus, the opener has no incentive to ever betcall, as he only hurts himself.

Now let us consider what happens when the opener does not bluff-checkfold. Again, half the time, the dealer will have the king and the other half, the dealer will have the ace. If checked to while holding the king, the dealer will check 100% of the time. Thus, the expected value is $\frac{1}{2}(-1)=-\frac{1}{2}$ if the dealer holds the king. The other half the time, the dealer will have the ace. If checked to, the dealer will betcall 100% of the time with the ace. In response, the opener will fold 100% of the time. Thus, the expected value from the dealer holding the ace if bet to is $\frac{1}{2}(-1)=-\frac{1}{2}$. Summing these two together, we see that the return from the opener checking the queen is -1.

We can again verify that checkcalling and checkraising are sub-optimal strategies. It is clear that if the dealer holds the king, what the opener plans to do after checking is a moot point. This is because the dealer will always force a face off by checking. However, these three post-checking strategies differ wildly when the dealer holds the ace. We know that the dealer will betcall 100% of the time with the ace. Calling or raising following the dealer's bet will simply bleed additional money from the opener. Thus, for the opener with the queen, checkcalling and checkraising are indeed dominated by checkfolding.



From this analysis, we clearly see that the strategy we have presented for the queen is an equilibrium strategy. There is no benefit to increasing the probability by which the dealer either bluffs or checks, as the return is -1 in both cases.

Opener King

We now move on to the king. The opener will always check with the king, but how he reacts after the dealer moves differs. $\frac{1}{3} + \frac{1-\alpha}{3}$ of the time, the opener will call. The other $\frac{2}{3} - \frac{1-\alpha}{3}$ of the time, the opener will fold.

We begin by analyzing the checkcall. Given that the opener checks with the king, $\frac{1}{2}$ of the time the dealer has the queen, and the other $\frac{1}{2}$ he has the ace.

With the queen, the dealer will check with $\frac{2}{3}$ probability, giving a $+1$ return to the opener. The other $\frac{1}{3}$, the dealer will betfold, which will prompt the opener to call. This gives $+2$ returns to the opener. Thus, if the opener king checks to the dealer's queen, the return will be $\frac{1}{2}[\frac{2}{3}(+1) + \frac{1}{3}(+2)] = \frac{1}{2}\frac{4}{3} = \frac{2}{3}$.

The other half the time, the dealer has the ace. With the ace, the dealer will betcall 100% of the time. Since the opener will now call 100% of the time, there is a -2 expected value. Thus, the expected value from this branch is $\frac{1}{2} * (-2) = -1$.

Summing the returns from these branches, we see that the returns from the opener checkcalling the king are $\frac{2}{3} - 1 = -\frac{1}{3}$.

We now move on to the right branch, which is the checkfold branch. Once the opener checks, half the time, the dealer will have the queen, and the other half, the ace.

If the dealer has the queen, $\frac{2}{3}$ of the time he will check and $\frac{1}{3}$ of the time he will betfold. If the dealer betfolds then the opener will subsequently fold, and lose -1. If the dealer checks, then the opener will win $+1$. Thus, the returns for the dealer checkfolding the king to the dealer's queen are $\frac{1}{2}[(\frac{2}{3}(+1) + \frac{1}{3}(-1)] = \frac{1}{2}\frac{1}{3} = \frac{1}{6}$.

The other half of the time, the dealer will have the ace. With the ace, the dealer will betcall 100% of the time. In this branch, the opener will subsequently fold, and thus lose -1. Thus, the returns for the opener checkfolding the king to the dealer's ace are $\frac{1}{2}(-1) = -\frac{1}{2}$.

Summing these returns, we see that the returns from checkfolding the king are $-\frac{1}{2}+\frac{1}{6}=-\frac{1}{3}$.

We see that the returns from checkcalling and checkfolding the king are exactly the same: $-\frac{1}{3}$.

While we have verified that the opener is indifferent between checkcalling and checkfolding, we must also verify that checkraising and betting are sub-optimal.

We first approach the strategy of checkraising. We know that if the opener checkraises with the king, the dealer will hold the queen and ace $\frac{1}{2}$ the time each.

With the queen, the dealer will check with $\frac{2}{3}$ probability and betfold with $\frac{1}{3}$ probability. If the dealer checks, the opener will win $+1$. Since this occurs with $\frac{2}{3}$ probability, the opener has expected value $+\frac{2}{3}$. If the dealer betfolds, then the opener will raise which will prompt a fold by the dealer. This occurs $\frac{1}{3}$ of the time and yields $+2$ for the opener, for an expected value of $\frac{2}{3}$. Overall, the opener checkraising the king against the dealer's queen will result in expected value $\frac{1}{2}(\frac{2}{3}+\frac{2}{3})=\frac{2}{3}$.

The other half the time, the dealer will have the ace. Once again, the dealer will betcall 100% of the time upon being checked to. Since the opener is checkraising, the opener will subsequently raise after the dealer's bet, and then the dealer will call. The opener loses -3 from this action. We see that checkraising the king against the dealer's ace gives expected value $-\frac{3}{2}$.

Overall, we can see that checkraising the king gives the opener expected value $\frac{2}{3}-\frac{3}{2}=\frac{4-9}{6}=-\frac{5}{6}$. This is clearly lower expected value than either checkcalling or checkfolding. Thus, the opener has direct incentive not to checkraise and instead stick with checkcalling and checkraising.

On the other hand, it is conceivable that the opener might want to bet the king. We now show that this is a poor choice.

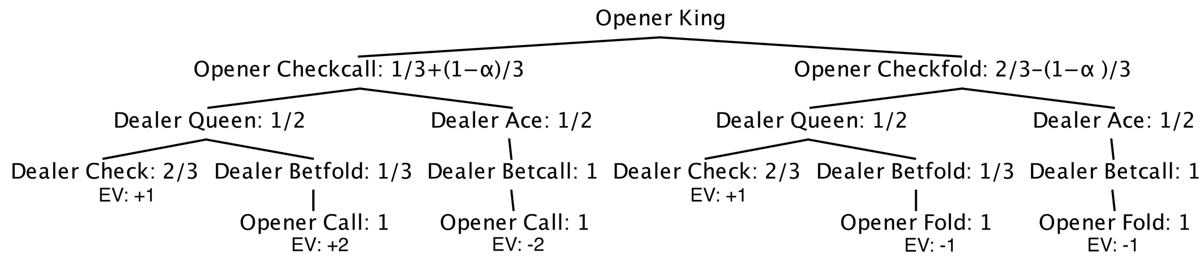
When the opener bets his king, $\frac{1}{2}$ the time the dealer holds the queen and $\frac{1}{2}$ the time the dealer holds the ace.

If the dealer holds the queen, then the dealer will always fold upon being bet to, giving the opener his ante of $+1$. Betting the king against the dealer's queen therefor gives the opener expected value of $\frac{1}{2}$.

If the dealer holds the ace, then the dealer will always raise upon being bet to. In the best case scenario where the opener immediately folds, the opener will lose -2. If the opener chooses to call the raise, then the opener will lose -3. Betfolding the king against the dealer's ace will yield an expected value of -1 and betcalling the king against the dealer's ace will yield an expected value of $-\frac{3}{2}$.

Summing the values from these two scenarios tells us that the opener has expected value $-\frac{1}{2}$ from betfolding his king and expected value -1 from betcalling his king. In both cases, the opener is significantly better off from checkfolding or checkcalling his king.

Therefor, the opener has no incentive to deviate from his optimal strategy of checkcalling with probability $\frac{1}{3}+\frac{1-\alpha}{3}$ and checkfolding with probability $\frac{2}{3}-\frac{1-\alpha}{3}$.



Opener Ace

Finally, we analyze the strategy for the ace. According to our solution, the opener will checkraise the ace with probability α and betcall with probability $1-\alpha$. Let us begin by considering the checkraise component.

If the opener checkraises with the ace, the dealer holds the queen and king half the time each respectively. With the queen, the dealer will check $\frac{2}{3}$ of the time, and betfold $\frac{1}{3}$ of the time. Checking will yield 1, while the betfold will result in the raise by the opener (and subsequent fold by the dealer) for a yield of +2. Thus, if the dealer has a queen, the value will be $\frac{1}{2}[\frac{2}{3}(+1)+\frac{1}{3}(+2)] = \frac{1}{2}\frac{4}{3} = \frac{2}{3}$.

The other half the time, the dealer will hold the king. With a king, the dealer will always check, yielding 1. Thus, if the dealer has a king, the yield will be $\frac{1}{2}(+1) = \frac{1}{2}$. Overall, checkraising will give expected value $\frac{1}{2}+\frac{2}{3} = \frac{7}{6}$.

If the opener betcalls with the ace, then again the dealer holds the queen and king with $\frac{1}{2}$ probability each. With the queen, the dealer will always fold for a yield of +1 and with the king, the dealer will fold for a yield of +1 $\frac{2}{3}$ of the time. The other $\frac{1}{3}$, the dealer will call for a yield of +2. Overall, we can see that the yield will be $\frac{1}{2}(1)+\frac{1}{2}[\frac{2}{3}(1)+\frac{1}{3}(2)] = \frac{1}{2}+\frac{1}{2}\frac{4}{3} = \frac{1}{2}+\frac{2}{3} = \frac{7}{6}$.

Again, as expected, we see that checkraising the ace will yield the same expected value as will betcalling the ace: $\frac{7}{6}$. However, we must now explore the opener's other potential strategies with the ace- checkfolding, checkcalling and betfolding- and verify that they do not hold higher expected value for the opener.

If the opener checkfolds with the ace, the dealer holds the queen and king half the time respectively.

With the queen, the dealer will check $\frac{2}{3}$ of the time, and betfold $\frac{1}{3}$ of the time against the check. If the dealer checks, the opener gains +1. If the dealer betfolds, then the opener loses -1. Expected value from the opener checkfolding the ace to the dealer's queen is $\frac{1}{2}(\frac{2}{3}-\frac{1}{3}) = \frac{1}{6}$.

With the king, the dealer will always check when checked to. This gives expected value $\frac{1}{2}(1) = \frac{1}{2}$. Overall, we can see that if the opener checkfolds his ace, his expected value is $\frac{2}{3}$, which is clearly lower than the $\frac{7}{6}$ expected value he could have earned had he checkraised or betcalled.

What about checkcalling? Once again, if the opener checkcalls his ace, then $\frac{1}{2}$ the time the dealer will have a queen and $\frac{1}{2}$ the time the dealer will have a king.

As above, the dealer will check his queen with $\frac{2}{3}$ probability and betfold with $\frac{1}{3}$ probability upon being checked to. The dealer's check yields the opener $+1$. The dealer's betfold will lead to the opener calling said bet and yield $+2$ for the opener. Expected value from the opener checkcalling his ace to the dealer's queen gives expected value $\frac{1}{2}(\frac{2}{3} + 2\frac{1}{3}) = \frac{2}{3}$.

The other half the time, the dealer will hold a king when the opener checkcalls his ace. When checked to, the dealer will always check his king. The dealer therefor donates his ante of $+1$ to the opener. Checkcalling the ace to the dealer's king will result in expected value $\frac{1}{2}(1) = \frac{1}{2}$.

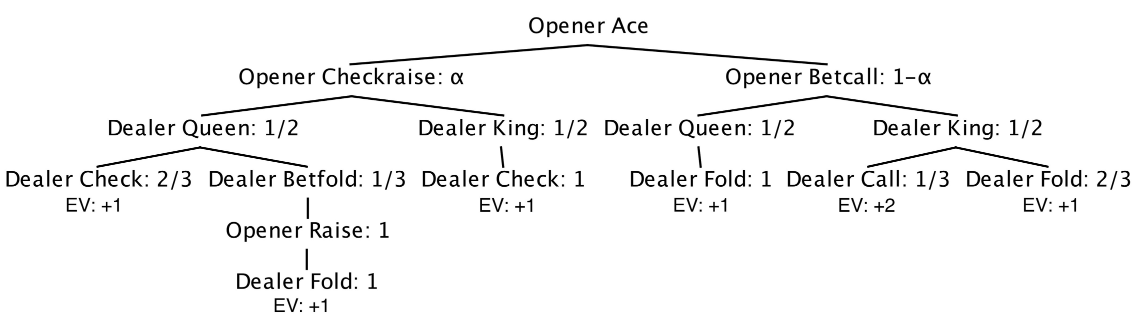
If we sum the expected values from checkcalling the ace to the dealer's queen and king, we get the expected value of the opener checkcalling the ace: $\frac{1}{2} + \frac{2}{3} = \frac{7}{6}$. This is certainly unexpected, although not surprising, as we will explore later. The important point is that the opener cannot improve his expected value by unilaterally shifting his strategy from either checkraising or betcalling the ace to checkcalling. However, the fact that checkcalling the ace has the same expected value as checkraising and betcalling points to another set of optimal strategies.

We now move to betfolding the ace. If the opener betfolds with the ace, half the time the dealer will hold the queen and the other half, the king.

With the queen, the dealer will always fold upon being bet to, giving the opener $+1$. Betfolding the ace to the dealer's queen will therefor give the opener expected value $\frac{1}{2}$.

With the king, the dealer will call with $\frac{1}{3}$ probability and fold with $\frac{2}{3}$ upon being bet to. If the dealer folds, then he gives the opener his ante of $+1$. If the dealer calls the bet, then he donates $+2$ to the opener. Expected value of betfolding is therefor $\frac{1}{2}(\frac{2}{3} + 2(\frac{1}{3}) = \frac{2}{3}$.

Overall, we can see that the expected value from the opener betfolding his ace is $\frac{1}{2} + \frac{2}{3} = \frac{7}{6}$. Again, this is an unexpected result, although not completely out of left field. However, since the expected value from betfolding the ace is the same as checkraising and betcalling (and checkcalling for that matter), the opener cannot improve his expected value for the game by unilaterally shifting his strategy in favor of betfolding the ace.



At this point, we have demonstrated that given any card, the strategies the opener can employ will all yield the same expected value (given the dealer's strategy). Thus, the opener has no incentive to deviate from the mixed strategy we have laid out in this section, and cannot improve his expected value by deviating from this strategy. Although we see that there are two alternative strategies for the opener with the ace that provide the same expected value, the opener has no real incentive to deviate towards these strategies. We will explore this in more depth in the discussion.

Proving Equilibrium for the Dealer

We have shown that the strategy above is an equilibrium strategy for the opener. We now show that the strategy we have found is an equilibrium strategy for the dealer. To do this, we show that given the dealer is dealt a given card and given the opener's initial move, each potential strategy has the same expected value. Additionally, we will show that each alternative strategy that the dealer assigns zero probability to in the strategy does in fact have lower expected value- although not necessarily strictly lower.

Dealer Queen

We begin with the queen. According to our strategy, the dealer will fold 100% of the time if he is bet to. Folding will obviously give expected value -1. Alternatively, if checked to, the dealer will check with $\frac{2}{3}$ probability and betfold with $\frac{1}{3}$ probability (assuming he is checked to).

If the dealer checks, then his expected value from checking is -1. This is obvious since the opener has either the king or ace, so a check would result in the dealer losing his ante.

If the dealer betfolds with the specified $\frac{1}{3}$ probability, then again the opener has the king or ace. However, the opener no longer has $\frac{1}{2}$ probability of having either the king or ace. Since the first move was a check, we must consider the probabilities that the opener has a king/ace given that he checked in the first round. We know that the opener will always check the king and check the ace with probability α. Thus, given that the opener checked the first round, there is $\frac{1}{1+\alpha}$ probability that the opener has a king and $\frac{\alpha}{1+\alpha}$ probability that the opener has an ace. With the ace, the opener will always raise. This results in an expected value of -2 $\frac{\alpha}{1+\alpha}$ of the time, for an expected yield of $-\frac{2\alpha}{1+\alpha}$. On the other hand, if the opener has the king, then the opener will fold $\frac{2}{3} - \frac{1-\alpha}{3}$ of the time and call $\frac{1}{3} + \frac{1-\alpha}{3}$ of the time (since the opener checkfolds with probability $\frac{2}{3} - \frac{1-\alpha}{3}$ and checkcalls with probability $\frac{1}{3} + \frac{1-\alpha}{3}$, which sums to 1). If the opener check folds, then the dealer wins 1, and if the opener checkcalls, then the dealer loses -2. This results in a yield of $\frac{1}{1+\alpha}[(\frac{2}{3} - \frac{1-\alpha}{3})(+1) + (\frac{1}{3} + \frac{1-\alpha}{3})(-2)] = \frac{1}{1+\alpha}(\frac{-3+3\alpha}{3}) = \frac{-1+\alpha}{1+\alpha}$.

Overall, we see that betfolding the queen will give an expected value of $-\frac{2\alpha}{1+\alpha} + \frac{-1+\alpha}{1+\alpha} = \frac{-1-\alpha}{1+\alpha} = -1$.

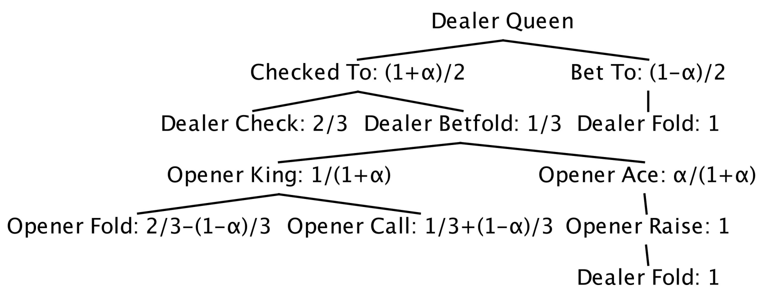
As we expect, both checking and betfolding the queen will result in an expected value of -1. We quickly verify that the dealer is no better off by betcalling if checked to and no better off calling or raising if bet to when he holds the queen.

With betcalling, we know that the above analysis for betfolding applies when the opener possesses the king. The opener always checkcalls or checkfolds with the king, so the second stage of the dealer's strategy when betting does not really matter- the game simply never gets to that stage. However, when the opener possesses the ace, things change. Since the opener will always raise the ace in response to the dealer's bet, calling the raise is strictly worse than folding the raise. Since betfolding provides the same expected value in most cases and strictly higher expected value in one case, betcalling is a dominated strategy.

Regarding calling, it's easy to see that the dealer will always lose if he calls the queen. Since he will lose -2 by calling instead of only -1 by folding, it is clear that the dealer will never call with the queen.

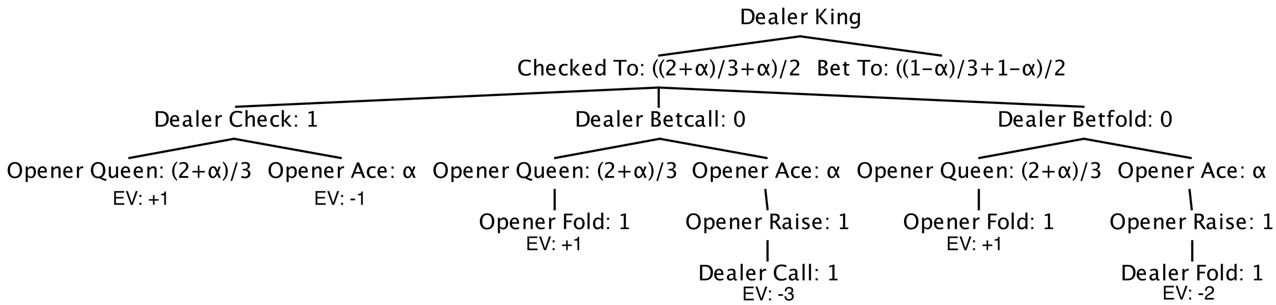
Finally, we look at what happens if the dealer raises with the queen after being bet to. We know that the opener either has the king or ace. Furthermore, the opener always checks if he holds the king, so if the opener bets and the dealer holds the queen, the dealer knows that the opener has the ace. The opener will call any raise that the dealer might attempt to bluff, which will lose the dealer -3. Clearly, this is strictly worse off than folding if bet to so the dealer will never raise his ace after being bet to.

As a result of this analysis, we see that the best strategies available to the dealer when he holds the queen are checking and betfolding when checked to, and folding when he is bet to. When checked to, he is indifferent between checking and betting and thus has no incentive to deviate from the prescribed strategy. When bet to, the dealer is significantly worse off if he calls or raises, so he will cut his losses and check.



Dealer King

We move on to the king. According to our strategy, the dealer will check the king 100% of the time if he is checked to. To see that this is a smart strategy, we know that if the dealer is checked to and holds the king, then $\frac{\frac{2+\alpha}{3}}{\frac{2+\alpha}{3}+\alpha}$ of the time, the opener holds the queen and the other $\frac{\alpha}{\frac{2+\alpha}{3}+\alpha}$ of the time, the opener holds the ace (since the opener will checkfold the queen $\frac{2+\alpha}{3}$ of the time and the opener will checkraise the ace α of the time). The opener will only check with the intention of folding if he holds the queen. Thus, regardless of whether the dealer checks, betcalls, or betfolds, the dealer will win $+1$ if the opener holds the queen and checks. On the other hand, the opener will always check with the intention of raising if he holds the ace. If the dealer checks against this, then the dealer loses -1. If the dealer bets with the intention of calling, then the opener will raise, dealer will call, and subsequently lose -3. Similarly, if the dealer bets with the intention of folding, the the opener will raise, dealer will fold, and subsequently lose -2. In these three cases, it is clear that it is preferable for the dealer to cut his losses and always check with the king if checked to.

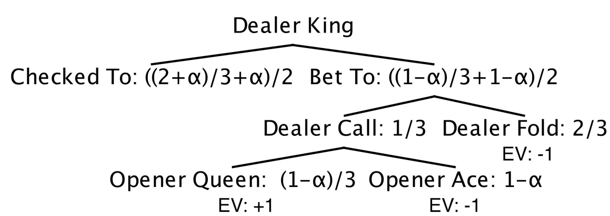


On the other hand, if the dealer is bet to and holds the king, he will call 1/3 of the time and fold 2/3 of the time. Let us first analyze the strategy of calling. Since the dealer has the king, the opener had either the ace or queen when betting. According to our family of solutions, we know that the opener will bet the ace $1-\alpha$ and that the opener will bet the queen $\frac{1-\alpha}{3}$. Thus, the dealer will earn $+2$ $(\frac{1}{4})$ and lose $-2\ \frac{3}{4}$ for calling. This results in -1.

If the dealer folds, then he will lose -1 regardless of what the opener has (by definition). Thus, we see that the dealer is indifferent between calling and folding if he is bet to and holds the king as the dealer will receive -1 either way.

We also verify that the the dealer is not incentivized to raise when holding the king if he is bet to. Assuming the dealer is bet to, $\frac{1}{4}$ of the time the opener holds the queen and $\frac{3}{4}$ of the time the opener holds the ace. If the opener holds the queen, the opener will fold the dealer's raise, giving the dealer $+2$. On the other hand, if the opener holds the ace, then the opener will call the dealer's raise, causing the dealer to lose -3. Overall, the expected value from the dealer raising the king when bet to is $\frac{1}{4}(+2)+\frac{3}{4}(-3)=\frac{2}{4}-\frac{9}{4}=-\frac{7}{4}$. Clearly this is a worse outcome than had the dealer folded or called.

We see that the dealer is indifferent between calling and folding, and strictly prefers these two over raising. He therefor has no incentive to deviate from his strategy of folding with probability $\frac{2}{3}$ and calling with probability $\frac{1}{3}$.



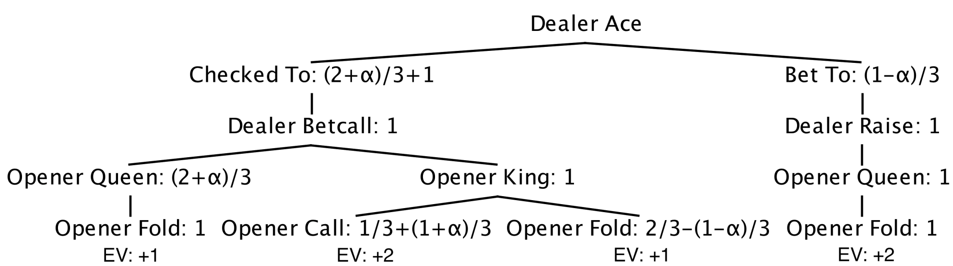
Dealer Ace

Moving on the ace, our strategy states that if the dealer is checked to, he will always betcall. Similarly, if he is bet against, he will always raise. We knew this to be the case when we stated earlier that all other strategies were "stupid mistakes." As a precaution, we will verify that all other strategies are indeed stupid mistakes.

First on our list is the dealer folding the ace after getting bet to. We know that raising the ace will always yield some positive expected value, regardless of what card the opener has. Folding the ace is indeed a dominated strategy.

What about the dealer calling his ace after getting bet to? If the dealer has the ace, then the opener has either the queen or king. We know that the opener never bets his king- so it must be true that the opener holds the queen and has decided to betfold. If the dealer decides to call the bet, then he wins +2. However, if the dealer decides to raise, then the opener will invariably fold. Once again, the dealer will win +2. It is therefor not clear that the dealer would decide not to call his ace. However, he has no incentive to call the ace, since he is not strictly better off doing so.

We now move on to the dealer betfolding the ace after getting checked to. Contingent upon the dealer holding the ace, the opener must have either the queen or king. Given that the opener checked in the first round, the opener intends to fold his queen if the dealer bets. Alternatively, the opener will sometimes call and sometimes fold his king if the dealer bets. However, regardless of what card the opener holds, the game will always end after the dealer bets upon being checked to. Once again, we see that the dealer is indifferent between betfolding and betcalling his ace- the game never progresses far enough where the dealer's move after betting matters. Importantly though, the dealer is no worse off by betcalling his ace, and therefor has no incentive to deviate from the strategy specified.



From this analysis, we see that the strategy we have specified for the dealer is indeed an optimal strategy. We have shown that there are deviations the dealer can make that would not decrease the expected value of the game for him. However, since the dealer cannot strictly improve his expected value via any deviations, the strategy remains an optimal one.

Expected Value of the Game

We have so far shown that the strategies presented form an equilibria, and that neither player can benefit himself by unilaterally deviating from the specified strategy. We now calculate the expected value of the game for both players. Since it is a two player zero sum game, we know that the expected value of the game to the opener must be the additive inverse of the expected value of the game to the dealer. However, we will independently show the expected values for both opener and dealer for verification purposes.

Opener Expected Value

We showed above that the expected value to the opener of the queen is -1. This arises since each of his optimal strategies has expected value -1, so regardless of how he chooses to choose between checking and betting, he will always have expected value -1. Similarly, we showed that the expected value to the opener of holding the king is $-\frac{1}{3}$ and the expected value of holding the ace is $\frac{7}{6}$.

Unfortunately for the opener, the expected value of this game is negative. Since the opener will be dealt the queen, king and ace each with $\frac{1}{3}$ probability, the overall expected return of the game for the opener is $\frac{1}{3}(-1-\frac{1}{3}+\frac{7}{6}) = \frac{1}{3}(-\frac{4}{3}+\frac{7}{6}) = \frac{1}{3}(-\frac{1}{6}) = -\frac{1}{18}$. Since this is a two-player zero-sum game, the expected return of the game for the dealer must be $+\frac{1}{18}$.

Dealer Expected Value

We know from computing the opener's expected value that the dealer's expected value is $\frac{1}{18}$. In this section, we verify this result.

Since the expected value from the dealer getting checked to and from getting bet to is -1, we can see that the expected value given that the dealer has a queen is -1. Since the dealer will hold a queen $\frac{1}{3}$ of the time, the expected value for a queen is $-\frac{1}{3}$.

We now compute the expected value of the dealer being dealt the king. We know that if he is bet to, he will lose -1. He is bet to $\frac{\frac{(1-\alpha)}{3}+1-\alpha}{2}$ of the time, so this branch has expected value $-\frac{2}{3}(1-\alpha)$. On the other hand, if the dealer is checked to, then he will win $+1$ $\frac{\frac{2+\alpha}{3}}{\frac{2+\alpha}{3}+\alpha}$ of the time and lose -1 $\frac{\alpha}{\frac{2+\alpha}{3}+\alpha}$ of the time. Since he is checked to $\frac{\frac{2+\alpha}{3}+\alpha}{2}$ of the time, the expected value of the branch is $\frac{\frac{2+\alpha}{3}+\alpha}{2}(\frac{\frac{2+\alpha}{3}}{\frac{2+\alpha}{3}+\alpha} - \frac{\alpha}{\frac{2+\alpha}{3}+\alpha}) = \frac{\frac{2+\alpha}{3}+\alpha}{2}(\frac{\frac{2-2\alpha}{3}}{\frac{2+\alpha}{3}+\alpha}) = \frac{1-\alpha}{3}$. If we sum the expected values of being checked to and being bet to with the king, we that the expected value for the dealer holding the king is $-\frac{2}{3}(1-\alpha)+\frac{1}{3}(1-\alpha) = -\frac{1}{3}(1-\alpha)$. Since the dealer will be dealt the king $\frac{1}{3}$ of the time, the expected value here is $-\frac{1}{9}(1-\alpha)$.

We now compute the expected value to the dealer of being dealt an ace. The dealer will be checked to $\frac{\frac{2+\alpha}{3}+1}{2}$ of the time and will be bet to the other $\frac{1-\alpha}{6}$ of the time. If the dealer is checked to, the dealer will always betcall. The opener's response depends on his card. $\frac{\frac{2+\alpha}{3}}{\frac{2+\alpha}{3}+1}$ of the time, the opener will have a queen. In such a case, the opener will always fold and give the dealer $+1$. The other $\frac{1}{\frac{2+\alpha}{3}+1}$ of the time, the opener will have a king. If the opener has the king, the opener will call the dealer's bet $\frac{1}{3}+\frac{1-\alpha}{3}$ of the time giving the dealer $+2$ or the opener will fold the dealer's bet $\frac{2}{3}-\frac{1-\alpha}{3}$ of the time giving the dealer $+1$. Overall, the value to the dealer of being checked to and holding the ace is: $\frac{\frac{2+\alpha}{3}+1}{2} * (\frac{\frac{2+\alpha}{3}}{\frac{2+\alpha}{3}+1} + (\frac{1}{\frac{2+\alpha}{3}}(2(\frac{1}{3}+\frac{1-\alpha}{3})+(\frac{2}{3}-\frac{1-\alpha}{3})))) = \frac{1}{2}(\frac{2+\alpha}{3}+\frac{2}{3}+\frac{2-2\alpha}{3}+\frac{2}{3}+\frac{-1+\alpha}{3}) = \frac{1}{2}(\frac{7}{3}) = \frac{7}{6}$.

On the other hand, if the dealer is bet to, then the dealer will always raise. The only way the opener will bet against the dealer's ace is with the queen- and he will always fold that queen (giving the dealer value $+2$). Thus, the expected value for the dealer being bet to with the ace is $\frac{2-2\alpha}{6}$.

If we sum the values from these two branches, we will get the expected value from the dealer holding an ace. This gives us $\frac{7}{6}+\frac{2-2\alpha}{6}=\frac{9-2\alpha}{6}$. Since the dealer will be dealt an ace $\frac{1}{3}$ of the time, the dealer has expected value $\frac{9-2\alpha}{18}$ for the ace.

If we sum the expected values for the dealer holding the queen, king and ace, we will get the dealer's expected value for this game. Summarizing the above, the dealer has expected value $-\frac{1}{3}$ for the queen, expected value $-\frac{1}{9}(1-\alpha)$ for the king and expected value $\frac{9-2\alpha}{18}$ for the ace. Summing these together yields $-\frac{1}{3}-\frac{1}{9}(1-\alpha)+\frac{9-2\alpha}{18}=\frac{-6}{18}+\frac{-2+2\alpha}{18}+\frac{9-2\alpha}{18}=\frac{1}{18}$. This is as we expected.

Discussion

Checkraising has sometimes been seen as a impolite play. According to professional poker player and TV star Phil Gordon, "[Check-raising] has endured some criticism over the years- some players of old believed that this particular type of duplicity constituted bad manners...[2]" That being said, checkraising- or sandbagging as it is sometimes called- is a valid and sometimes very effective strategy in poker. However, it seems that this is not the case here.

As we showed above, what either player does with the ace is almost inconsequential. For the opener, checkcalling, checkraising, betcalling and betfolding the ace all give expected values of $+\frac{7}{6}$. Similarly, the dealer is indifferent between between betfolding or betcalling the ace when checked to and calling or raising when bet to. If the opener wanted to be politely rational and avoid sandbagging while retaining the same expected value, he could shift to checkcalling.

Why is this? We showed very early on that each of the strategies that surprised us was in fact a dominated strategy. However, they were only strictly dominated under the condition of a raise. If a raise were never to be called, then raising and calling appear the same, regardless of what round of betting it is. Indeed, this is the case in our game.

However, it should be noted that both players actually should not play the so-called "dominated strategies." It is entirely true in the equilibrium that these strategies give the same expected value as do the "optimal strategies." However, there is a non-zero probability that one player will slip up and deviate from optimality. In such a case, checkraising may produce better results than checkcalling. Similar arguments apply for each pair of optimal vs. dominated strategies. Essentially, our definition of dominant strategies stands, and it is only in this precarious equilibria where raises do not get called that certain strategies are not strictly dominated.

It is also an interesting coincidence that the expected value of the game is $-\frac{1}{18}$ for the opener and $\frac{1}{18}$ for the dealer. These numbers seem extremely familiar. Indeed, they are the same expected values for Kuhn poker. If we look closely at the solution set for both the dealer and opener, we notice that it very closely resembles that for Kuhn poker. In fact, the trees look identical. That is, including the possibility of a raise has not done anything to change the probabilities found in the simpler game.

This is a fairly surprising result- we would imagine that the introduction of a raise might help tip the odds in favor of one player. Admittedly, it was never clear which player the possibility of raising helps (and it turns out that raising actually helps neither player).

However, it was easily conceivable that raising might help further a bluff, or that it would allow either player to effectively sandbag.

All of this harkens back to the raise's lack of utility. Since the appearance of a raise always triggers a fold by the subsequent player, we know that no matter how many raises are allowed, the expected value of the game would not change for either player. Moreover, it is easy to see that the strategy presented above is an equilibrium strategy for any number of available raises.

Admittedly, this is only true in the equilibrium. We have shown above that neither player has an incentive to deviate unilaterally from the strategy given. However, if one spots that the opponent has deviated by mistake, then it is to that player's advantage to also deviate. In such circumstances, it is conceivable that one could win more or less than the expected $+/-\frac{1}{18}$ we have found using this strategy. However, one always trends back toward this strategy in order to minimize losses. As theorized, deviations from equilibrium will drift back towards equilibrium.

We would again like to raise the point that this family of solutions may not be the only set of Nash Equilibrium strategy solutions. However, we do know that the expected values for any equilibrium strategy must be $\frac{1}{18}$ and $-\frac{1}{18}$ for the dealer and opener respectively. Imagine an equilibrium strategy in which the expected value of the game of the opener were $< -\frac{1}{18}$. We know that the strategies described above in this paper have an expected value of $-\frac{1}{18}$... so the opener could deviate unilaterally and achieve $-\frac{1}{18}$ expected value. However, this contradicts the idea of a mixed strategy Nash Equilibrium, that a player cannot deviate unilaterally and increase his expected value of the game. This argument also negates the possibility of a Nash Equilibrium strategy which gives expected value $> -\frac{1}{18}$ for the opener. This argument applies for both the game with only one raise as well as the game with more than one raise.

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