

# Interval Vector Polytopes

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# Abstract

An *interval vector* is a  $\{0, 1\}$ -vector where all the ones appear consecutively. An *interval vector polytope* is the convex hull of a set of interval vectors in  $\mathbb{R}^n$ . We study several classes of interval vector polytopes which exhibit interesting combinatorial-geometric properties. In particular, we study a class whose volumes are equal to the catalan numbers, another class whose legs always form a lattice basis for their affine space, and a third whose face numbers are given by the pascal 3-triangle.

# Chapter 1

## Introduction

An *interval vector* is a  $\{0, 1\}$ -vector in  $\mathbb{R}^n$  such that the ones (if any) occur consecutively. These vectors can nicely and discretely model scheduling problems where they represent the length of an uninterrupted activity, and so understanding the combinatorics of these vectors is useful in optimizing scheduling problems of continuous activities of certain lengths.

Much of this work is first done in a paper on which I was a cowriter of at the MSRI-UP in 2012.[BDDPR]

In this thesis we consider interval vectors as geometric points in  $\mathbb{R}^n$  and our goal is to understand their geometric relationship to each other. To do this we take sets of interval vectors as vertices of objects called polytopes, which are  $n$ -dimensional generalizations of 2d polygons or 3d polyhedra. We call a polytope constructed from interval vectors an *interval vector polytope*, introduced in [Da]. In essence, this project catalogues different interval vector polytopes and proves some elegant and interesting properties about them.

In chapter 3, we first consider the polytope formed by taking the convex hull of every interval vector in  $\mathbb{R}^n$  (section 3.1), and notice that it has volume  $C_n$  the  $n$ -th

catalan number. The surprising fact that this polytope, named the *complete interval vector polytope* is a catalanotope is interesting on its own, but what is more fascinating is that many of its subpolytopes (generated by taking smaller sets of interval vectors in  $\mathbb{R}^n$ ) also have very interesting structure to them.

For example, in section 3.2 we consider the *fixed interval vector polytope* which is the polytope whose vertices are all the interval vectors in a dimension with a fixed interval length (i.e. the same number of ones). This polytope turns out to be a unimodular simplex (which is the generalization of a unit triangle). This is interesting because a unimodular simplex is the 'smallest' polytope possible whose vertices are all integer valued, and it can tessellate to fill the entire plane hitting each lattice point. Essentially its legs form a basis for the lattice points of the plane.

Finally, we consider the *interval pyramid*, which is the polytope whose vertices are the standard unit vectors in  $\mathbb{R}^n$ , plus the interval vectors with one 0. Though this may seem like an arbitrary selection of interval vectors as vertices, we still see the interval pyramid exhibiting fascinating properties. It's volume is very easily calculated to be  $2(n-2)$  and its face numbers (the number of faces the polytope has in each dimension) reflect a combinatorial sequence called the pascal 3-triangle, which is like Pascal's triangle summed with a shifted Pascal's triangle, and thus is a sum of binomial coefficients. <sup>3</sup> While each of these results are at least mildly interesting, together, they seem to say something quite deep. Essentially, the geometric locations of the interval vectors on the plane are such that, even seemingly arbitrary sets of interval vectors create beautiful combinatorial-geometric objects. Somehow a number of famous combinatorial sequences are contained within the geometric relationships between interval vectors, making them geometric objects of note.

The second chapter of this thesis introduces the necessary terminology and basic theorems necessary to study convex polytopes and their combinatorial structure.

Much of my preliminary research in the field of convex polytopes and their combinatorics in general can be found in [Gr] and [Zi], and a for lattice polytopes and Ehrhart theory, in [Be]. Many definitions are pulled from these sources, which would be good to consult for a more complete introduction to the field.

In this second chapter we define the convex hull, the affine subspace, and introduce lattice polytopes and the basics of Ehrhart theory. We also introduce posets and explain how they represent the face structure of a polytope, so that we can define duality of a polytope. Finally we introduce basic graph theory for use in certain proofs later in the paper. Those already familiar with this material may skip to chapter 3, though a brief skim of the first chapter may familiarize the reader with the notation we use.

The third chapter introduces interval vector polytopes and proves basic results about the entire class of polytopes. We then present the results on the complete and fixed interval vector polytopes. The entire fourth chapter is dedicated to studying the face structure, volume and duality of the interval pyramid. The final chapter presents some open questions and suggests future work for further study.

# Chapter 2

## Preliminaries

### 2.1 Convex Polytopes

The definitions in this section are adapted from [Gr], [Zi], and [Be]. Intuitively, given a set of vertices in  $\mathbb{R}^2$  we can 'connect the dots' to form a polygon. This idea is precisified and generalized to  $\mathbb{R}^n$  with the following definition.

**Definition 1.** *If  $A = \{v_1, \dots, v_k\} \subset \mathbb{R}^n$ , we define the convex hull of  $A$  to be the set of nonnegative linear combinations of elements of  $A$  whose coefficients sum to 1.*

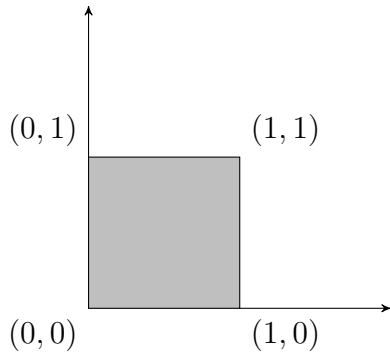
$$\text{conv}(A) := \left\{ \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_k v_k \mid \lambda_1, \lambda_2, \dots, \lambda_k \in \mathbb{R}_{\geq 0} \text{ and } \sum_{i=1}^k \lambda_i = 1 \right\}.$$

**Definition 2.** *A convex polytope  $\mathcal{P} \subseteq \mathbb{R}^n$  is the convex hull of finitely many points in  $\mathbb{R}^n$ .*

**Example 1.** *Let  $V = \{(0, 0), (0, 1), (1, 0), (1, 1)\} \subset \mathbb{R}^2$ . Then  $\text{conv}(V) = I \times I$  is the unit square in  $\mathbb{R}^2$ .*

We can envision a polytope in  $\mathbb{R}^n$  might have less than  $n$  dimensions (for example,





**Figure 2.1:**  $\text{conv}(V)$ .

a square embedded in 3 space). Such an object still spans a subspace of its entire ambient space which we call the polytope's *affine space*.

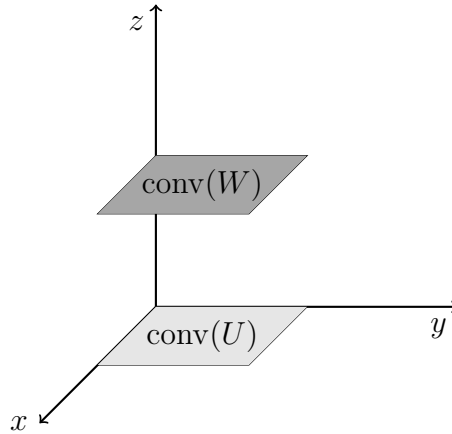
**Definition 3.** Let  $A = \{v_1, v_2, \dots, v_k\} \subset \mathbb{R}^n$ . The affine hull of  $A$  is defined as the set of linear combinations of elements of  $A$  whose coefficients sum to 1.

$$\text{aff}(A) := \left\{ \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_k v_k \mid \sum_{i=1}^k \lambda_i = 1 \right\}.$$

This is similar to the definition of the convex hull, without the restriction that all the coefficients are nonnegative. The affine hull of a set corresponds to the affine subspace of  $\mathbb{R}^n$  spanned by the convex polytope generated by that set.

**Example 2.** Let  $U = \{(0, 0, 0), (0, 1, 0), (1, 0, 0), (1, 1, 0)\} \subset \mathbb{R}^3$ . Then  $\text{conv}(U) \subset \mathbb{R}^3$  is the unit square lying flat on the  $xy$  plane and  $\text{aff}(U)$  is the  $xy$  plane itself,  $\{x \in \mathbb{R}^3 \mid x_3 = 0\}$ . Notice  $\text{aff}(U)$  is a 2 dimensional vector space.

**Example 3.** Let  $W = \{(0, 0, 1), (0, 1, 1), (1, 0, 1), (1, 1, 1)\} \subset \mathbb{R}^3$ . Then  $\text{conv}(W)$  is the unit square hovering at the plane  $\{x_3 = 1\} \subset \mathbb{R}^3$ . Then  $\text{aff}(W)$  is exactly the plane  $\{z = 1\} \subset \mathbb{R}^3$ . We notice that this affine subspace is not a linear subspace, but can still be viewed as a 2 dimensional vector space with addition defined as  $(a, b, 1) + (c, d, 1) =$



**Figure 2.2:** Non full dimensional polytopes.

$(a + c, b + d, 1)$ .

We would like a polytope to have a unique vertex set, but notice that various sets could have the same convex hull, so the vertices of a polytope  $\mathcal{P} = \text{conv}(\{v_1, \dots, v_k\}) \subset \mathbb{R}^n$  is not well defined as just  $\{v_1, \dots, v_k\}$ . Instead we must find a minimal set which generates  $\mathcal{P}$ .

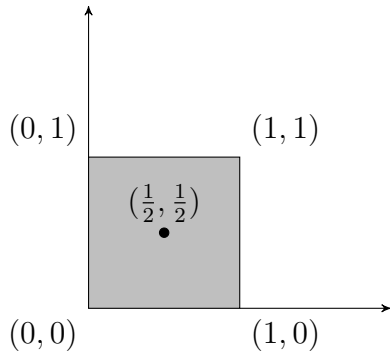
**Definition 4.** We call a set of points convexly independent if each point is not in the convex hull of the rest. That is,  $A$  is convexly independent if for all  $v \in A$ ,  $v \notin \text{conv}(A \setminus \{v\})$ .

The following proposition allows us to uniquely define the vertex set of a polytope.

**Proposition 1.** [Gr] Let  $A, B \subseteq \mathbb{R}^n$  be two convexly independent sets. If  $\text{conv}(A) = \text{conv}(B)$ , then  $A = B$ .

This justifies us in defining the vertex set as follows.

**Definition 5.** Let  $\mathcal{P} \subseteq \mathbb{R}^n$  be a convex polytope. The vertex set of  $\mathcal{P}$ , denoted  $\text{vert}(\mathcal{P})$  is defined as the set of convexly independent points whose convex hull is  $\mathcal{P}$ .



**Figure 2.3:** Convexly dependent point in  $B$ .

**Example 4.** The set  $B = \{(0, 0), (0, 1), (1, 0), (1, 1), (1/2, 1/2)\}$  is not convexly independent because  $(1/2, 1/2) = \frac{1}{2}(0, 1) + \frac{1}{2}(1, 0)$  so that  $(1/2, 1/2) \in \text{conv}(A \setminus \{(1/2, 1/2)\})$ . The set  $A = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$  is convexly independent, and  $\text{conv}(A) = \text{conv}(B)$ . The vertex set of the polytope  $\text{vert}(\text{conv}(B)) = A$ .

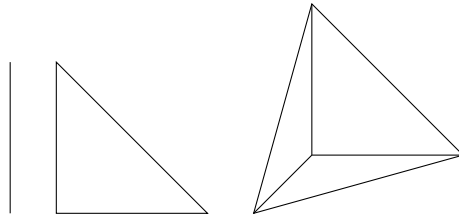
We can similarly define affine independence of a set.

**Definition 6.** A set of points is called *affinely independent* if each point is not in the affine hull of the rest. That is,  $A$  is affinely independent if for all  $v \in A$ ,  $v \notin \text{aff}(A \setminus \{v\})$ .

Now that we have uniquely defined vertices, we can begin to discuss the dimension of a convex polytope  $\mathcal{P}$ .

**Definition 7.** We define the *affine space* of a convex polytope to be the affine hull of its vertices. We can always view the affine space of dimension  $d$  of a polytope as a vector space, and we define the *dimension* of  $\mathcal{P} \subset \mathbb{R}^n$  to be the dimension of its affine space. We denote this  $\dim(\mathcal{P})$ , and we call  $\mathcal{P}$  a  $d$ -polytope. If  $d = n$ , then  $\mathcal{P}$  is full dimensional.

Notice that the polytopes described in examples 2 and 3 are 2 dimensional polytopes embedded in 3 dimensional space, so they are not full dimensional. On the other



**Figure 2.4:** 1, 2, and 3 dimensional simplices

hand, the polytope described in example 1 is full dimensional. A very important class of  $d$ -dimensional polytopes are those with exactly  $d + 1$  vertices.

**Definition 8.** *A  $d$  dimensional polytope with exactly  $d + 1$  vertices is called a  $d$ -simplex.*

Simplexes are very important, because they have a very uniform structure, and their volumes are very easy to compute (in fact, I will define volume using simplexes). Most importantly, any polytope can be *triangulated*, that is can be expressed as the union of simplexes. Since the volume of each simplex is easy to compute, a triangulation of a convex polytope makes the volume easy to compute.

**Example 5.** *A line segment is a 1-simplex. A triangle is a 2-simplex. A tetrahedron is a 3-simplex.*

## 2.2 Lattice Polytopes

Most of the polytopes we consider are those with integral vertices, called *lattice polytopes*. All examples considered so far are lattice polytopes.

**Definition 9.** *A point  $p \in \mathbb{Z}^n$  is called a lattice point of  $\mathbb{R}^n$ . A lattice polytope is a polytope whose vertices are all lattice points.*

**Definition 10.** [Be] Let  $\mathcal{A} \subseteq \mathbb{R}^n$  be a  $d$ -dimensional affine subspace. A set  $\{v_1, \dots, v_d\}$  of  $d$  integer valued vectors in  $\mathbb{R}^n$  is said to be a lattice basis for  $\mathcal{A}$  if, fixing a lattice point  $p \in \mathcal{A}$ , any other lattice point  $q \in \mathcal{A}$  can be expressed as an integral linear combination of the lattice basis, plus  $p$ . That is any lattice point  $q \in \mathcal{A}$  can be written as

$$q = p + \lambda_1 v_1 + \dots + \lambda_d v_d$$

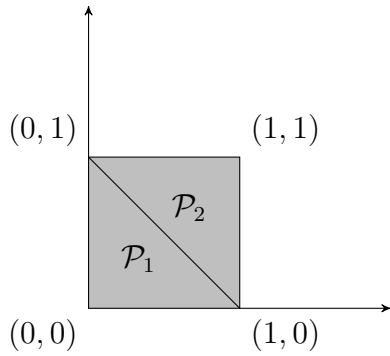
with each  $\lambda_i \in \mathbb{Z}$ . The lattice basis of an affine subspace  $\mathcal{A}$  is essentially a basis for the lattice points of  $\mathcal{A}$  viewed as a vector space with origin  $p$ .

**Definition 11.** We call a  $d$ -simplex  $\mathcal{P}$  with vertex set  $\{v_1, \dots, v_{d+1}\}$  to be a unimodular  $d$ -simplex, if the legs  $\{v_2 - v_1, v_3 - v_1, \dots, v_{d+1} - v_1\}$  form a lattice basis for  $\text{aff}(\mathcal{P})$ .

**Example 6.** Let  $\mathcal{P} = \text{conv}(\{(0, 0), (0, 1), (1, 0)\})$ . Then the legs of this polytope are  $(1, 0)$  and  $(0, 1)$  which form a lattice basis for  $\mathbb{R}^2$ . So  $\mathcal{P}$  is a unimodular simplex.

We can now assign a *normalized volume* of 1 to any unimodular simplex, so that for any arbitrary  $d$ -polytope  $\mathcal{P}$ , we define its *normalized volume* with respect to unimodular  $d$ -simplexes having volume 1. Thus, we can triangulate  $\mathcal{P}$  into unimodular  $d$ -simplexes, and the number of simplexes in such a triangulation is the normalized volume of  $\mathcal{P}$ , denoted  $\text{vol}(\mathcal{P})$ . It is known that this definition of volume is well-defined.[Be]

**Example 7.** If  $\mathcal{P} = \text{conv}(\{(0, 0), (0, 1), (1, 0), (1, 1)\})$ , then we can triangulate it via  $\mathcal{P}_1 = \text{conv}(\{(0, 0), (0, 1), (1, 0)\})$  and  $\mathcal{P}_2 = \text{conv}(\{(0, 1), (1, 0), (1, 1)\})$ . Both  $\mathcal{P}_1$  and  $\mathcal{P}_2$  are unimodular, and  $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$  so that the  $\text{vol}(\mathcal{P}) = 2$ .



**Figure 2.5:** Triangulation of the unit square.

**Proposition 2.** [Da] Let  $\mathcal{P} \in \mathbb{R}^n$  be a full dimensional  $n$ -simplex with vertex set  $\{v_0, \dots, v_n\}$ . Then:

$$\text{vol}(\mathcal{P}) = \det(v_1 - v_0, v_2 - v_0, \dots, v_n - v_0).$$

This determinant is called the Cayley-Menger determinant.

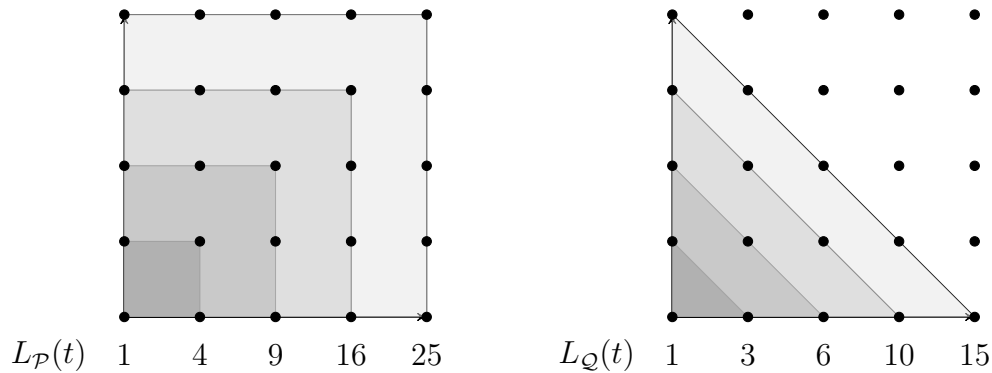
Notice that we would expect the volume of the 2-cube to be 1. It turns out that for a  $d$ -polytope  $\mathcal{P}$  the the normalized volume of  $\mathcal{P}$  is  $d!$  times the volume of  $\mathcal{P}$ .

### 2.2.1 Ehrhart Theory

How a polytope behaves when it is dilated or contracted can tell us important things about its combinatorial structure. For  $t \in \mathbb{Z}_{\geq 0}$ , the  $t$ -dilation of a polytope  $\mathcal{P}$  is

$$t\mathcal{P} := \{tv \mid v \in \mathcal{P}\}.$$

Each lattice  $d$ -polytope  $\mathcal{P}$  has associated with it an *Ehrhart polynomial*, denoted  $L_{\mathcal{P}}(t)$ . When  $t \in \mathbb{Z}_{\geq 0}$ , the polynomial yields the number of lattice points in the  $t^{\text{th}}$  dilate of the polytope. It is known that the constant term of any Ehrhart polynomial



**Figure 2.6:** The Ehrhart polynomials of the unit triangle and unit square produce the triangle numbers and square numbers respectively.

is 1, and that the degree of this polynomial is the dimension  $d$  of  $\mathcal{P}$ . Importantly, the leading coefficient of the Ehrhart polynomial is its volume, i.e.  $\frac{1}{d!} \text{vol}(\mathcal{P})$ . [Be]

**Example 8.** Let  $\mathcal{P} = \text{conv}(\{(0, 0), (0, 1), (1, 0), (1, 1)\})$ . Then  $L_{\mathcal{P}}(t) = (t+1)^2$ , which are the square numbers as we would expect. Notice that the leading coefficient is 1, therefore the volume of the square is 1, and the normalized volume is 2.

**Example 9.** Let  $\mathcal{Q} = \text{conv}(\{(0, 0), (0, 1), (1, 0)\})$ . Then  $L_{\mathcal{Q}}(t) = \frac{(t)(t+1)}{2}$  which are the triangular numbers as we would expect. We notice that the leading coefficient is  $\frac{1}{2}$ , therefore the volume of the triangle is  $\frac{1}{2}$  and the normalized volume is 1 as desired (since the triangle is unimodular).

Call a transformation *lattice preserving* if it takes a lattice basis to a lattice basis. Linear, lattice preserving bijections between polytopes preserve combinatorial structure in the following way.

**Proposition 3.** [Be] A linear, lattice preserving bijection between polytopes preserves the Ehrhart polynomial. Thus if  $\mathcal{P}$  and  $\mathcal{Q}$  are polytopes, and  $T : \mathcal{P} \rightarrow \mathcal{Q}$  is a linear lattice preserving bijection, then  $L_{\mathcal{P}}(t) = L_{\mathcal{Q}}(t)$ .

## 2.3 Faces, Face Lattices and the Dual

A lot of the interesting combinatorial data in a convex polytope lives within its face structure. That is, the way faces of different dimensions intersect can often have very interesting combinatorial properties. A question we hope to answer in chapter 4, is whether the interval pyramid is self-dual. In this section we define the necessary terminology to begin to try and solve this problem. We first rigorously define the inequality description of a polytope and use that to define the faces of a polytope, stating basic but important theorems. We then briefly review posets and define the face lattice of a polytope as a poset. Next we define the dual form of a polytope and define what it would mean for a polytope to be self-dual.

### 2.3.1 Faces of a Polytope

The sequence of definitions and results in this section are adapted from [Zi]. Proofs of all these results can be found there.

**Definition 12.** Let  $\mathcal{P} \subseteq \mathbb{R}^n$ . If  $c \in \mathbb{R}^n$ , we define the linear equality  $c \cdot x \leq c_0$  to be valid for  $\mathcal{P}$  if it is satisfied for all points  $x \in \mathcal{P}$ .

**Definition 13.** A face  $F$  of a  $d$ -polytope  $\mathcal{P} \subset \mathbb{R}^n$  is any set of the form:

$$F = \mathcal{P} \cap \{x \in \mathbb{R}^n : c \cdot x = c_0\}$$

where  $c \cdot x \leq c_0$  is a valid inequality. A face of a polytope is a polytope, and thus has dimension. We call a face  $F$  a  $k$ -face of  $\mathcal{P}$  if it has dimension  $k$ . A 0-face is a vertex, a 1-face is called an edge, and a  $(d - 1)$ -face is called a facet. If  $\mathcal{P}$  has  $f_k$



$k$ -faces, we define the  $f$ -vector of  $\mathcal{P}$  to be

$$f(\mathcal{P}) := (f_0, f_1, \dots, f_{d-1})$$

It turns out than any polytope defined by vertices can be equivalently defined by it's  $n - 1$  dimensional faces, in what is called a *facet description* or *inequality description* of the polytope. This is summarized in the following theorem.

**Theorem 1.** (See [Zi] theorem 1.1).  $P \subset \mathbb{R}^n$  is a convex polytope if and only if it can be described as

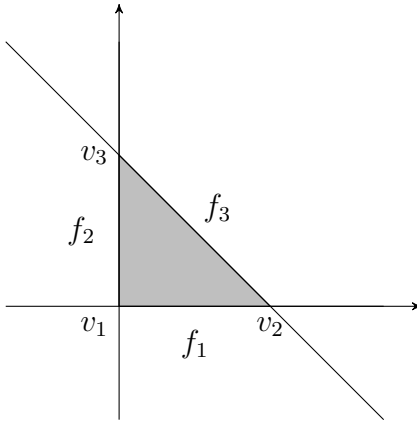
$$P = \{x \in \mathbb{R}^n : c_i \cdot x \leq d_i \text{ for some } c_1, \dots, c_m \in \mathbb{R}^n \text{ and } d_1, \dots, d_m \in \mathbb{R}\}$$

This is the unique facet description exactly when each  $c_i \cdot x \leq d_i$  is a valid inequality describing an  $n - 1$  dimensional face.

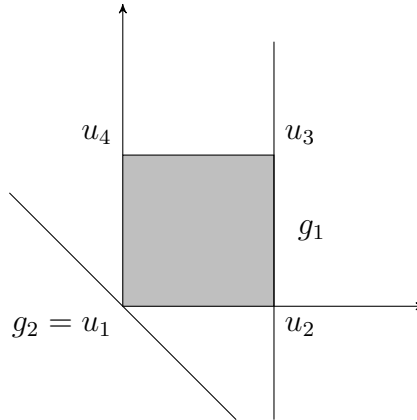
**Example 10.** The triangle  $\mathcal{P} = \text{conv}(\{(0, 0), (0, 1), (1, 0)\}) \subset \mathbb{R}^2$  has 3 vertices  $v_1 = (0, 0)$ ,  $v_2 = (0, 1)$ , and  $v_3 = (1, 0)$ . It also has three faces  $f_1 = \{x = 0\} \cap \mathcal{P}$ ,  $f_2 = \{y = 0\} \cap \mathcal{P}$ , and  $f_3 = \{x + y = 1\} \cap \mathcal{P}$ . Thus  $\mathcal{P}$  is a 2-dimensional polytope with 3 vertices and 3 edges has  $f$ -vector  $f(\mathcal{P}) = (3, 3)$ , and its facet description is

$$\mathcal{P} = \{c \in \mathbb{R}^2 : c \cdot (-1, 0) \leq 0, c \cdot (0, -1) \leq 0, c \cdot (1, 1) \leq 1\}.$$

**Example 11.** Consider the unit 2-cube  $\mathcal{P} = \text{conv}(\{(0, 0), (0, 1), (1, 0), (1, 1)\})$ . Name the vertices  $u_1, u_2, u_3, u_4$  respectively. We see that the linear inequality  $x_1 \leq 1$  is valid for  $\mathcal{P}$ . So  $g_1 = \mathcal{P} \cap \{x \in \mathbb{R}^2 \mid x_1 = 1\}$  is a 1 dimensional face of the 2-cube. Similarly,  $-(x_1 + x_2) \leq 0$  is a valid inequality, and  $g_2 = \mathcal{P} \cap \{x \in \mathbb{R}^2 \mid x_1 + x_2 = 0\} = \{(0, 0)\} = u_1$  is a zero dimensional face (or vertex) of the 2-cube. The 2-cubes facet description



**Figure 2.7:** The faces of the unit triangle.



**Figure 2.8:** Some faces of the unit square.

is

$$\mathcal{P} = \{c \in \mathbb{R}^2 : c \cdot (-1, 0) \leq 0, c \cdot (0, -1) \leq 0, c \cdot (1, 0) \leq 1, c \cdot (0, 1) \leq 1\}.$$

Notice that for any  $d$ -polytope  $\mathcal{P}$ , the inequality  $0 \cdot x \leq 0$  is valid, and  $0 \cdot x = 0$  for all  $x \in \mathcal{P}$ , so that  $\mathcal{P}$  is itself a  $d$ -face of  $\mathcal{P}$ . Similarly, the inequality  $0 \cdot x \leq 1$  is valid, but  $0 \cdot x \neq 1$  for any  $x \in \mathcal{P}$ . Thus  $\emptyset$  is also always a face of  $\mathcal{P}$  (defined to have dimension  $-1$ ). What follows is an important fact about the face structure of a

polytope.

**Proposition 4.** (See [Zi] proposition 2.3). (See for example: Ziegler Lectures on Polytopes Proposition 2.3). Let  $\mathcal{P} \subseteq \mathbb{R}^n$  be a polytope, and  $V$  its vertex set. Let  $F$  be a face of  $\mathcal{P}$ .

1.  $F$  is a polytope, with vertex set  $F \cap V$ .
2. Every intersection of faces of  $\mathcal{P}$  is a face of  $\mathcal{P}$ .
3. The faces of  $F$  are exactly the faces of  $\mathcal{P}$  that are contained in  $F$ .
4.  $F = P \cap \text{aff}(F)$ .

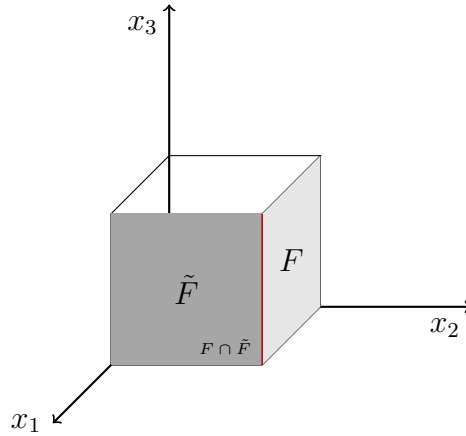
**Example 12.** Consider the 3-cube  $C_3 = \text{conv}(V)$  where  $V = \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (1, 0, 0), (0, 1, 1), (1, 1, 0), (1, 0, 1), (1, 1, 1)\}$ . We can consider the inequality  $x_1 \leq 1$  which is valid for  $C_3$  and defines the face  $F = C_3 \cap \{x \in C_3 : x_1 = 1\}$ . We know that this must be a polytope, and we know its vertex set is  $V' = F \cap V = \{(1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}$ . Thus  $F$  is the unit square lying flat on the plane  $\{x_1 = 1\}$ .

Next consider the inequality  $x_2 \leq 1$  which is valid for  $C_3$  and the face  $\tilde{F} = C_3 \cap \{x \in C_3 : x_2 = 1\}$ , with vertex set  $\tilde{V} = \tilde{F} \cap V = \{(0, 1, 0), (0, 1, 1), (1, 1, 0), (1, 1, 1)\}$  so that  $\tilde{F}$  is the unit square lying flat on the plane  $\{x_2 = 1\}$ .

We know that  $F \cap \tilde{F}$  must be a face of  $C_3$  as well, and its vertex set is  $\{(1, 1, 0), (1, 1, 1)\}$ , so that  $F \cap \tilde{F}$  is the line connecting those two points. It is also a face of  $C_3$  defined by the valid inequality  $x_1 + x_2 \leq 2$ .

### 2.3.2 The Face Lattice of a Polytope

We earlier introduced the concept of an  $f$ -vector of a polytope, which tells us the number of faces of each dimension that a convex polytope may have. In fact, we can



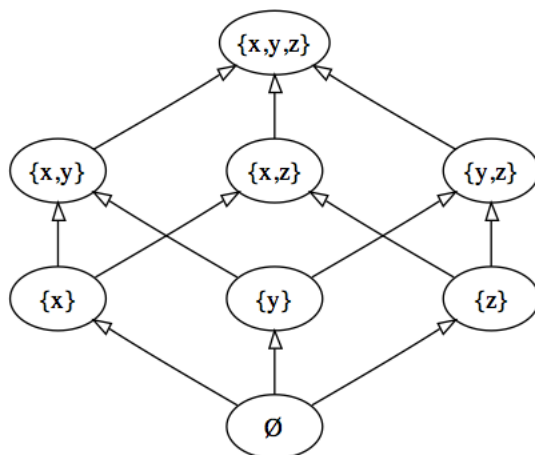
**Figure 2.9:** Some faces of the unit 3-cube.

consider a more structured way to consider the faces of a polytope, using a poset called a face lattice. Some terminology must first be introduced.

**Definition 14.** [Zi] A poset  $(S, \leq)$  is a finite set  $S$  equipped with a relation ' $\leq$ ' which satisfies the following properties.

1. Reflexivity: for all  $x \in S$ ,  $x \leq x$ .
2. Transitivity: if  $x \leq y$  and  $y \leq z$  then  $x \leq z$ .
3. Antisymmetry: if  $x \leq y$  and  $y \leq x$  then  $x = y$ .

For two elements  $x, y \in S$ , if  $x \leq y$  or  $y \leq x$  then we say  $x$  and  $y$  are *comparable*. A *maximal* element  $x \in S$  is an element such that there is no  $y \in S$  such that  $x \leq y$ . A *minimal* element  $y \in S$  is an element such that there is not  $x \in S$  such that  $x \leq y$ . A poset is *bounded* if there is a unique maximal element and unique minimal element. A chain (of length  $n$ ) is a list of elements  $x_1, \dots, x_n \in S$  such that  $x_1 \leq \dots \leq x_n$ . A bounded poset is *graded* if every maximal chain has the same length. For a poset  $S$ , we define the *opposite poset*  $S^{op}$  to have the same underlying set as  $S$ , with the relation  $x \leq_{op} y$  being in  $S^{op}$  if and only if  $y \leq x$  holds in  $S$ . We define two posets



**Figure 2.10:** The poset  $(P(\{x, y, z\}), \subseteq)$ . [Ks]

$(S, \leq_S), (T, \leq_T)$  to be isomorphic, denoted  $(S, \leq_S) \cong (T, \leq_T)$  if there is a bijection  $f : S \rightarrow T$  such that for all  $x, y \in S$ , it holds that  $x \leq_S y$  if and only if  $f(x) \leq_T f(y)$ .

**Example 13.** *The power set of  $\{x, y, z\}$  ordered by inclusion is the graded poset  $(P(\{x, y, z\}), \subseteq)$ , of length 4. An example of a maximal chain is  $\emptyset \subset \{z\} \subset \{y, z\} \subset \{x, y, z\}$ . Notice also that the opposite poset  $P\{x, y, z\}^{op}$  is isomorphic to  $P\{x, y, z\}$ .*

The face structure of a polytope can be nicely coded into posets in the following way.

**Definition 15.** *Let  $\mathcal{P}$  be a  $d$ -polytope whose faces are the set  $\mathcal{F}$ . Then we define the face lattice of a convex polytope  $\mathcal{P}$  to be  $(\mathcal{F}, \subseteq)$ , all the faces of  $\mathcal{P}$  ordered by inclusion, and call this poset  $L(\mathcal{P})$ . The length of  $L(\mathcal{P})$  is  $d + 2$ .*

**Definition 16.** *Two polytopes  $\mathcal{P}, \mathcal{Q} \subset \mathbb{R}^n$  are defined to be combinatorially isomorphic if their face lattices are isomorphic as posets, i.e.  $L(\mathcal{P}) \cong L(\mathcal{Q})$ .*

**Example 14.** *Two  $n$ -simplices are always combinatorially isomorphic, with face lattices equivalent to the poset of the power set  $n + 1$  elements, ordered by inclusion.*

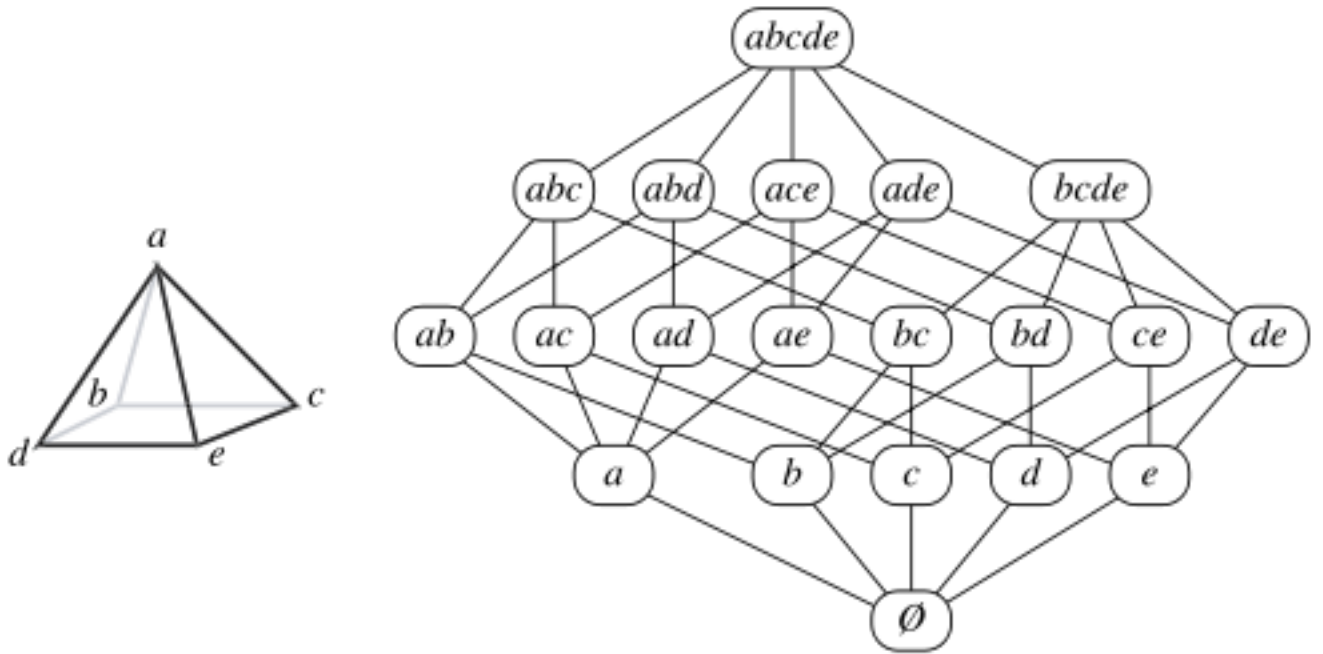


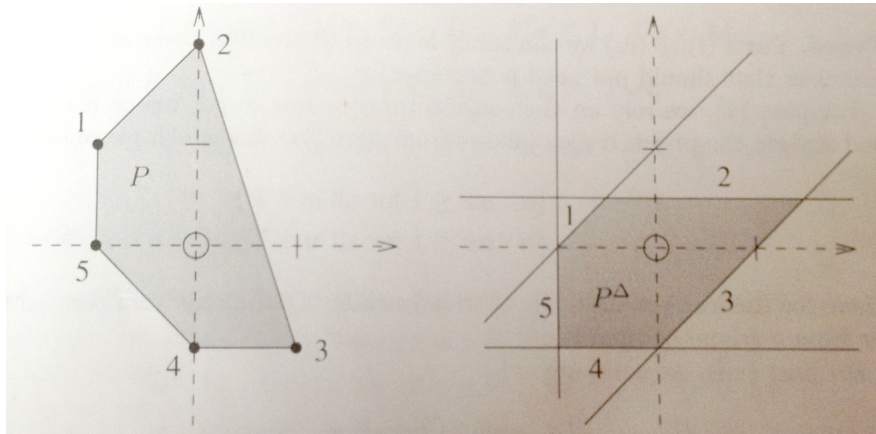
Figure 2.11: The face lattice of a square pyramid.[Ep]

### 2.3.3 Duality

A natural construction to take for a polytope is to consider its dual. Intuitively, this is turning a polytope inside-out, replacing each facet with a vertex and reconstructing the polytope from there. An interesting class of polytopes are those which are self-dual, meaning that the dual has the same combinatorial structure as the original polytope. We make these notions more precise here.

**Definition 17.** For a  $d$ -polytope  $\mathcal{P}$ , we define an interior point  $y \in \mathcal{P}$  to be a point which is not contained in a face of  $\mathcal{P}$  of dimension smaller than  $d$ . The collection of interior points of  $\mathcal{P}$  is called  $\text{int } \mathcal{P}$ .

We hope to define the dual of a polytope  $\mathcal{P}$ , but for the construction we use, we must have that  $0 \in \text{int } \mathcal{P}$ . This can generally be achieved for any polytope (as long as the polytope has nonempty interior) by a simple affine translation of the polytope



**Figure 2.12:** Taking the dual.[Zi]

so that 0 is in the interior.[Zi]

**Lemma 1.** ([Zi] Lemma 2.8). *Let  $\mathcal{P}$  be a polytope in  $\mathbb{R}^n$ . If  $p \in \mathcal{P}$  can be represented as  $p = \frac{1}{n+1} \sum_{i=0}^n x_i$  for  $n + 1$  affinely independent points  $x_0, \dots, x_n \in \mathcal{P}$ , then  $p$  is an interior point of  $\mathcal{P}$ .*

The following construction is from [Zi].

**Definition 18.** *Let  $\mathcal{P} \subseteq \mathbb{R}^n$  be a polytope with 0 in the interior. The dual of  $\mathcal{P}$  is defined by*

$$\mathcal{P}^\Delta := \{y \in \mathbb{R}^n : y \cdot x \leq 1 \text{ for all } x \in \mathcal{P}\},$$

where  $y \cdot x = y_1x_1 + \dots + y_nx_n$  is the dot product.

Figure 2.3.3 shows a convex pentagon on a plane given by its five vertices, and it's dual given by five inequalities.

**Definition 19.** *A polytope  $\mathcal{P} \subset \mathbb{R}^n$  with 0 in the interior is said to be self-dual if  $L(\mathcal{P}) \cong L(\mathcal{P}^\Delta)$ .*

**Proposition 5.** ([Zi] Corollary 2.14) *Let  $\mathcal{P} \subset \mathbb{R}^n$  be a polytope with 0 in the interior. The face lattice of  $\mathcal{P}^\Delta$  is the opposite poset of  $\mathcal{P}$ . That is  $L(\mathcal{P}^\Delta) = (L(\mathcal{P}))^{op}$ .*

**Corollary 1.** *A polytope  $\mathcal{P} \subset \mathbb{R}^n$  with 0 in the interior is self-dual if and only if  $L(\mathcal{P}) \cong L((\mathcal{P}))^{op}$ .*

The following proposition confirms the idea that our defined notion of taking the dual of a polytope in fact it replaces facets with vertices and so on.

**Proposition 6.** *Let  $\mathcal{P} \subseteq \mathbb{R}^n$  be a polytope with 0 in the interior, and vertex set  $\{v_1, \dots, v_m\}$ . If  $c \in \mathbb{R}^n$ , then  $c \in \mathcal{P}^\Delta$  if and only if  $c \cdot v_i \leq 1$  for  $i = 1, \dots, m$ .*

*Proof.* First if  $c \in \mathcal{P}^\Delta$ , then clearly  $c \cdot v_i \leq 1$  for all  $i = 1, \dots, m$  since  $c \cdot x \leq 1$  must be a valid inequality for all  $x \in \mathcal{P}$  (by how we defined the dual construction).

Conversely, assume  $c \cdot v_i \leq 1$  holds for each  $i$ . To show  $c \in \mathcal{P}^\Delta$  we must show that  $c \cdot x \leq 1$  for all  $x \in \mathcal{P}$ . Fix some  $x \in \mathcal{P}$ . We know that  $x = \lambda_1 v_1 + \dots + \lambda_m v_m$  with  $\sum_i \lambda_i = 1$  and each  $\lambda_i \geq 0$ . But then since  $c \cdot v_i \leq 1$  for each  $i$ , then  $\lambda_i(c \cdot v_i) \leq \lambda_i$  for each  $i$ . Let us sum  $\lambda_i(c \cdot v_i) \leq \lambda_i$  for each  $i$  so that

$$\lambda_1(c \cdot v_1) + \dots + \lambda_m(c \cdot v_m) \leq \lambda_1 + \dots + \lambda_m = 1$$

Clearly  $\lambda_i(c \cdot v_i) = c \cdot (\lambda_i v_i)$  so that

$$c \cdot (\lambda_1 v_1) + \dots + c \cdot (\lambda_m v_m) \leq 1$$

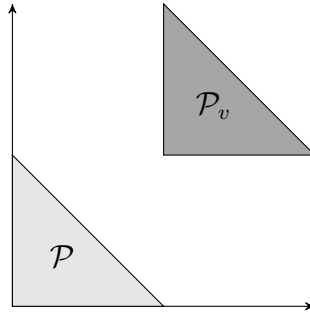
$$c \cdot (\lambda_1 v_1 + \dots + \lambda_m v_m) \leq 1$$

$$c \cdot x \leq 1$$

Since  $x$  was arbitrary,  $c \cdot x \leq 1$  for all  $x \in \mathcal{P}$ , so that  $c \in \mathcal{P}^\Delta$ . □

**Definition 20.** *We define an affine translation of a polytope  $\mathcal{P} \subset \mathbb{R}^n$  to be just a translation of all the points in the polytope by a single vector  $v \in \mathbb{R}^n$ . We denote the*





**Figure 2.13:** Affine translation of the unit triangle.

*translated polytope*

$$\mathcal{P}_v = \mathcal{P} + v$$

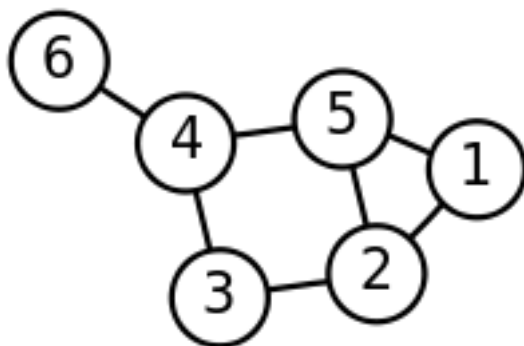
**Example 15.** Let  $\mathcal{P} = \text{conv}(\{(0,0), (1,0), (0,1)\})$  and  $v = (1,1)$ . Then  $\mathcal{P}_v = \text{conv}(\{(1,1), (2,1), (1,2)\})$ .

Notice that for any polytope  $\mathcal{P} \subset \mathbb{R}^n$  with nonempty interior, if  $v \in \text{int}(\mathcal{P})$ , then  $0 \in \text{int}(\mathcal{P}_{-v})$ . Thus it is not difficult to have an affine translation of  $\mathcal{P}$  with 0 on the interior. It is important to notice that *affine translation of a polytope does not affect the polytopes face lattice poset*[Zi]. In fact, it does not affect any of the combinatorial data of the polytope, and so we are justified in calling  $\mathcal{P}_{-v}^\Delta$  *the dual* of  $\mathcal{P}$ . Therefore, we can restate Proposition 19 for the more general case of polytopes with nonempty interiors.

**Corollary 2.** [Zi] Let  $\mathcal{P} \subset \mathbb{R}^n$  be a polytope with nonempty interior, and let  $v \in \text{int}(\mathcal{P})$ . Then,

$$L(\mathcal{P}_v^\Delta) = (L(\mathcal{P}))^{op}$$

Thus any  $\mathcal{P}$  is self dual if and only if  $L(\mathcal{P}) = (L(\mathcal{P}))^{op}$ .



**Figure 2.14:** An example of a graph.

## 2.4 Basic Graph Theory

Relating to any interval vector polytope, there is associated a graph called the *flow dimension graph* which tells us the affine-dimension of the polytope.[Da] I will introduce the necessary basics of graph theory and wait until later to define the flow dimension graph.

**Definition 21.** A graph is an ordered pair  $G = (V, E)$  where  $V$  is a set of vertices, and  $E \subseteq V \times V$  is a set of edges between pairs of vertices.

**Example 16.** Consider the graph  $G = (V, E)$  with six nodes, where  $V = \{1, 2, 3, 4, 5, 6\}$ , and the edge set  $E = \{(1, 2), (1, 5), (2, 3), (3, 4), (4, 5), (4, 6)\}$ . Each element of  $E$  can be visualized as an edge connecting the two vertices.

**Definition 22.** Two nodes  $a, b$  in a graph  $G = (V, E)$  are said to be connected if there exists a path from  $a$  to  $b$ , that is there exist  $q_0, \dots, q_s \in V$  such that  $(a, q_0), (q_0, q_1), \dots, (q_s, b) \in E$ . A connected component of a graph is a maximal set of vertices which are all connected.

Notice that the poset in example 16 has one connected component.

# Chapter 3

## Interval Vector Polytopes

We begin by defining interval vectors and interval vector polytopes in general. We then consider several classes of interval vector polytopes and the interesting combinatorial properties that arise within them.

**Definition 23.** [Da] A  $\{0, 1\}$ -vector is a vector whose entries are all in the set  $\{0, 1\}$ . An interval vector is a  $\{0, 1\}$ -vector in  $\mathbb{R}^n$  such that the ones (if any) occur consecutively. More precisely, a vector  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$  for  $i < k$ , if  $x_i = x_k = 1$ , then  $x_j = 1$  for all  $i \leq j \leq k$ . For  $i \leq j$ , we denote the  $\alpha_{i,j} := e_i + \dots + e_j$  where each  $e_k$  is the  $k$ th standard unit vector in  $\mathbb{R}^n$ . We define the interval length of  $\alpha_{i,j}$  to be  $j - i + 1$ , namely, the number of 1s appearing in the vector.

**Example 17.** Each standard unit vector is an interval vector with interval length 1.  $(0, 1, 1, 1, 0, 0) = \alpha_{2,4} \in \mathbb{R}^6$  is an interval vector with interval length 3.  $(1, 1, 1, 1) = \alpha_{1,4} \in \mathbb{R}^4$  is an interval vector with interval length 4. The 0 vector is always trivially an interval vector with interval length 0.  $(1, 0, 0, 1)$  is not an interval vector, since there are 0s between the first and fourth coordinates (both 1s).  $(0, 2, 2, 0)$  is not an interval vector because it is not a  $\{0, 1\}$ -vector.

**Definition 24.** A lattice polytope whose vertices consist entirely of interval vectors is called an interval vector polytope. If  $I \subseteq \mathbb{R}^n$  is a set of interval vectors, then  $\mathcal{P}_n(I) := \text{conv}(I)$  is the corresponding interval vector polytope.

**Example 18.**  $\mathcal{P}_2(\{(0, 0), (0, 1), (1, 0)\})$ , which is the triangle we have been considering in many examples, is an interval vector polytope. Similarly,  $\mathcal{P}_2(\{(0, 0), (0, 1), (1, 0), (1, 1)\})$  which is the unit 2-cube is also an interval vector polytope.

Let  $[n] := \{0, 1, \dots, n\}$ . Then for any set  $S \subseteq [n]$ , we define  $I_S = \{\text{interval vectors with interval length } t : t \in S\}$ . Define  $\mathcal{P}_n(I_S)$ , as the interval vector with vertices of interval length in  $S$ . This is a useful notational device to allow us to easily define interval vector polytopes coming from interval vectors of specific lengths.

**Example 19.** Let  $S = \{2\} \subseteq [4]$ . Then  $\mathcal{P}_4(I_S) = \text{conv}(\{(1, 1, 0, 0), (0, 1, 1, 0), (0, 0, 1, 1)\})$ . If  $T = \{1, 2\} \subseteq [3]$  then  $\mathcal{P}_3(I_T) = \text{conv}(\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (0, 1, 1)\})$ . We will look further at polytopes which are the convex hull of all interval vectors in  $\mathbb{R}^n$ :  $\mathcal{P}_n(I_{[n]}) = \text{conv}\{0, e_1, e_2, \dots, \alpha_{1,n}\}$ .

Notice that the vertex set of an interval vector polytope is not difficult to describe.

**Lemma 2.** Let  $I \subset \mathbb{R}^n$  be a set of interval vectors. Then  $I$  is convexly independent.

*Proof.* Since a subset of a convexly independent set is convexly independent, it suffices to show that the set of all interval vectors in  $\mathbb{R}^n$ ,  $I_{[n]}$ , is convexly independent. Consider  $\alpha_{i,j}$  where  $i$  can but does not need to equal  $j$ . Each other interval vector either has a 0 in the entry where  $\alpha_{i,j}$  has a 1, or a 1 where  $\alpha_{i,j}$  has a 0.

If a vector of the first type (with a 0 in the  $k$ th entry that  $\alpha_{i,j}$  has a 1) were to have nonzero coefficient, then the sum of the rest of the coefficients would be less than 1. But then, since no interval vector has a value greater than 1 in the  $k$ th entry,

the  $k$ th entry of any convex combination would be less than 1, and thus certainly the sum could not be  $\alpha_{i,j}$ .

If a vector of the second type (with a 1 in the  $k$ th entry where  $\alpha_{i,j}$  has a 0) were to have nonzero coefficient, then the  $k$ th entry of any convex combination would be greater than 0, and thus the sum could certainly not be  $\alpha_{i,j}$ . Thus  $\alpha_{i,j} \notin \text{conv}(I_{\{[n]\}} \setminus \{\alpha_{i,j}\})$ , and  $I_{\{[n]\}}$  is convexly independent, along with all of its subsets.  $\square$

**Corollary 3.** *Let  $I \subset \mathbb{R}^n$  be a set of interval vectors. Then the  $\text{vert}(\mathcal{P}_n(I)) = I$ .*

### 3.1 The Complete Interval Vector Polytope

We call  $\mathcal{P}_n(I_{[n]})$  the *Complete Interval Vector Polytope*. We compute the normalized volumes of the first few of these polytopes using polymake [Ga] and noticed a pattern. Let us look at a few examples.

**Example 20.**  $\mathcal{P}_1(I_{[1]}) = \text{conv}(\{0, 1\}) = [0, 1] \subset \mathbb{R}$  which is the unit interval.

$$\text{vol}(\mathcal{P}_1(I_{[1]})) = 1.$$

**Example 21.**  $\mathcal{P}_2(I_{[2]}) = \text{conv}(\{(0, 0), (0, 1), (1, 0), (1, 1)\}) \subset \mathbb{R}^2$  which is the unit 2-cube.

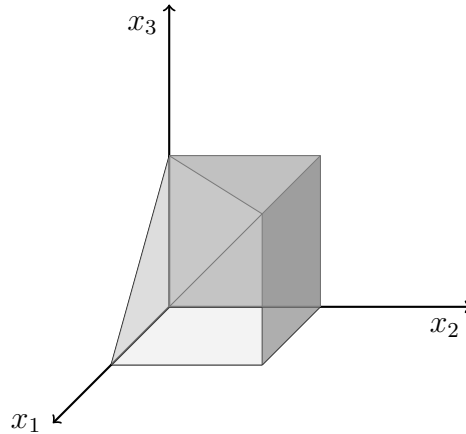
$$\text{vol}(\mathcal{P}_2(I_{[2]})) = 2.$$

**Example 22.**  $\mathcal{P}_3(I_{[3]}) = \text{conv}(\{e_1, e_2, e_3, \alpha_{1,2}, \alpha_{2,3}, \alpha_{1,3}\}) \subset \mathbb{R}^3$ . This looks like the unit 3-cube with a corner cut out and  $\text{vol}(\mathcal{P}_3(I_{[3]})) = 5$ .

**Example 23.**  $\mathcal{P}_4(I_{[4]}) = \text{conv}(\{e_1, e_2, e_3, e_4, \alpha_{1,2}, \alpha_{2,3}, \alpha_{3,4}, \alpha_{1,3}, \alpha_{2,4}, \alpha_{1,4}\}) \subset \mathbb{R}^4$ .

$$\text{vol}(\mathcal{P}_4(I_{[4]})) = 14.$$

An important sequence of numbers that arises often in combinatorics is called the *Catalan numbers*,  $\{C_n\}_{n=1}^{\infty}$ , defined by



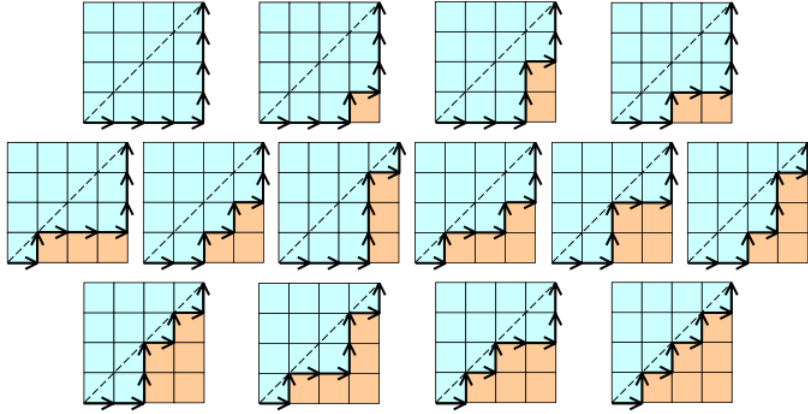
**Figure 3.1:** The third complete interval vector polytope,  $P_3(I_{[3]})$ .

$$C_n = \frac{1}{n+1} \binom{2n}{n} \quad (3.1)$$

These numbers are often the solutions to counting problems and represent, among other things, the number of monotonic paths between opposite corners in an  $n \times n$  grid not crossing above the line  $y = x$ . We find this example in [St], along with over 200 occurrences of the catalan numbers in an appendix. The catalan numbers are fascinating because of their frequent appearances in different combinatorial problems.

We notice that the first four numbers in the sequence are 1, 2, 5, 14, and some computation shows the first 10 interval vector polytopes have Catalan volume.

In his paper [Po], Postnikov defines the *complete root polytope*  $Q_n \subseteq \mathbb{R}^n$  as the convex hull of 0 and  $e_i - e_j$  for all  $i < j$ , and shows that the volume of  $Q_n$  is  $C_{n-1}$ . As it turns out, we can provide a lattice preserving linear bijection between  $Q_n$  and  $\mathcal{P}_{n-1}(I_{[n-1]})$ . By proposition 3, this implies the two polytopes have the same Erhart polynomial, and since the first term of the Erhart polynomial is the volume of the polytope, the two must share volumes.



**Figure 3.2:** The fourth catalan number represented as monotonic paths in a  $4 \times 4$  grid.[Dm]

**Theorem 2.**  $L_{Q_n}(t) = L_{\mathcal{P}_{n-1}(I_{[n-1]})}$ .

I will first present the proof by providing a lattice preserving linear bijection between the two polytopes. To illustrate how it works, I then compute the transformation on a series of examples.

*Proof.* Each of the vertices of  $Q_n$  are vectors with entries that sum to zero, so any linear combination (and specifically any convex combination) of these vertices also has entries who sum to zero:

$$\sum_i x_i = \sum_j y_j = 0 \implies \sum_i ax_i + \sum_j by_j = a \sum_i x_i + b \sum_j y_j = 0.$$

Define  $B := \{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = 0\}$ ; then  $Q_n \subset B$ .  $B$  is an  $(n - 1)$ -dimensional affine subspace of  $\mathbb{R}^n$ .

Consider the linear transformation  $T$  given by the  $n \times n$  lower triangular  $(0, 1)$ -matrix where  $t_{ij} = 1$  if  $i \geq j$  and  $t_{ij} = 0$  otherwise. Then the image

$$T(B) \subseteq \{x \in \mathbb{R}^n \mid x_n = 0\} =: A.$$

$A$  is also an  $n - 1$  dimensional subspace of  $\mathbb{R}^n$ . Since  $T$  has determinant 1, it is injective when restricting the domain to  $B$ . For the same reason, we know that for any  $y \in A$ , there exists  $x \in \mathbb{R}^n$  such that  $y = T(x)$ . But since  $y_n = \sum_{i=1}^n x_i = 0$ , then  $x \in B$ , so that  $T|_B : B \rightarrow A$  is surjective, and therefore a linear bijection.

Now consider the projection  $\pi : A \rightarrow \mathbb{R}^{n-1}$  given by

$$\pi((x_1, \dots, x_{n-1}, 0)) = (x_1, \dots, x_{n-1}).$$

The transformation is clearly linear, and has the inverse

$$\pi^{-1}((x_1, \dots, x_{n-1})) = (x_1, \dots, x_{n-1}, 0),$$

so that  $\pi$  is a bijection.

Now we show that the linear bijection  $\pi \circ T|_B : B \rightarrow \mathbb{R}^{n-1}$  is a lattice-preserving map. First we find a lattice basis for  $B$ . Consider the set

$$S := \{e_{i,n} = e_i - e_n \mid i < n\}.$$

We notice that any integer point of  $B$  can be represented as

$$\left( a_1, \dots, a_{n-1}, -\sum_{i=1}^{n-1} a_i \right) = \sum_{i=1}^{n-1} a_i e_{i,n}.$$

Any integer point of  $B$  is an integer combination elements of  $S$ , so  $S$  is a lattice basis of  $B$ .

Note that  $\pi \circ T(e_{i,n}) = e_i + \dots + e_{n-1} =: u_i$ . Therefore

$$\pi \circ T(S) = \{u_i \mid i \leq n - 1\} =: U.$$



We notice that  $e_{n-1} = u_{n-1}$  and  $e_i = u_i - u_{i+1}$ , so that each of the standard unit vectors  $e_i$  of  $\mathbb{R}^{n-1}$  is an integral combination of the vectors in  $U$ . Since the standard basis is a lattice basis of  $\mathbb{R}^{n-1}$ , so is  $U$ , thus  $\pi \circ T|_B$  is a lattice-preserving map.

Since our bijection is linear and lattice-preserving, all we have left to show is that the vertices of  $Q_n$  map to those of  $\mathcal{P}_{n-1}(I_{[n-1]})$ . By linearity,  $\pi \circ T(0) = 0$ , and given any vertex  $\alpha_{i,j}$  for  $\mathcal{P}_{n-1}(I_{[n-1]})$ , we know that  $\pi \circ T(e_{i,j+1}) = \alpha_{i,j}$  where  $i < j + 1 \leq n$  so that  $\pi \circ T|_B$  maps vertices to vertices.  $\square$

I'll illustrate how this works in the following examples.

**Example 24.** *Consider the third complete root polytope,*

$$Q_3 = \text{conv}(\{0, e_1 - e_2, e_2 - e_3, e_1 - e_3\}) = \text{conv}(\{(0, 0, 0), (1, -1, 0), (1, 0, -1), (0, 1, -1)\}).$$

*Since the transformation is linear, it preserves convex combinations (since they are just linear combinations), so we need only notice that it takes vertices to vertices.*

$$\pi(T(0, 0, 0)) = \pi(0, 0, 0) = (0, 0) = 0,$$

$$\pi(T(1, -1, 0)) = \pi(1, 0, 0) = (1, 0) = e_1,$$

$$\pi(T(1, 0, -1)) = \pi(1, 1, 0) = (1, 1) = \alpha_{1,2},$$

$$\pi(T(0, 1, -1)) = \pi(0, 1, 0) = (0, 1) = e_2.$$

*These are exactly the vertices  $P_2(I_{[2]})$ .*

**Example 25.** *Consider the third complete interval vector polytope*

$$P_3(I_{[3]}) = \text{conv}(\{(0, 0, 0), (0, 0, 1), (0, 1, 0), (1, 0, 0), (0, 1, 1), (1, 1, 0), (1, 1, 1)\}).$$

Transforming the vertices we see

$$T^{-1}(\pi^{-1}(0, 0, 0)) = T^{-1}(0, 0, 0, 0) = (0, 0, 0, 0) = 0,$$

$$T^{-1}(\pi^{-1}(0, 0, 1)) = T^{-1}(0, 0, 1, 0) = (0, 0, 1, -1) = e_3 - e_4,$$

$$T^{-1}(\pi^{-1}(0, 1, 0)) = T^{-1}(0, 1, 0, 0) = (0, 1, -1, 0) = e_2 - e_3,$$

$$T^{-1}(\pi^{-1}(1, 0, 0)) = T^{-1}(1, 0, 0, 0) = (1, -1, 0, 0) = e_1 - e_2,$$

$$T^{-1}(\pi^{-1}(0, 1, 1)) = T^{-1}(0, 1, 1, 0) = (0, 1, 0, -1) = e_2 - e_4,$$

$$T^{-1}(\pi^{-1}(1, 1, 0)) = T^{-1}(1, 1, 0, 0) = (1, 0, -1, 0) = e_1 - e_3,$$

$$T^{-1}(\pi^{-1}(1, 1, 1)) = T^{-1}(1, 1, 1, 0) = (1, 0, 0, -1) = e_1 - e_4.$$

These are exactly the vertices of  $Q_4$ .

A corollary to this theorem describes the normalized volume of the complete interval vector polytope.

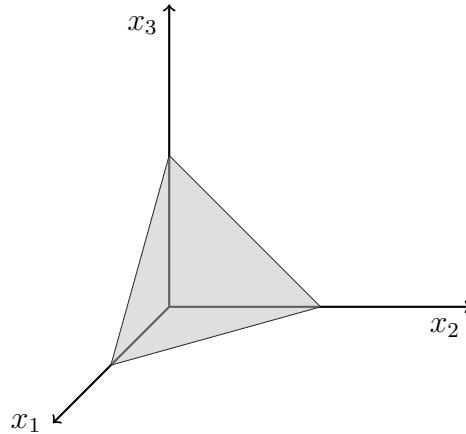
**Corollary 4.**  $\text{vol}(P_n(I_{[n]})) = C_n = \frac{1}{n+1} \binom{2n}{n}.$

## 3.2 The Fixed Interval Vector Polytope

Given an interval length  $i$  and a dimension  $n$ , we define the *fixed interval vector polytope*  $\mathcal{P}_n(I_{\{i\}}) \subset \mathbb{R}^n$  as the convex hull of all interval vectors in  $\mathbb{R}^n$  with interval length  $i$ .

**Example 26.** *The fixed interval vector polytope with  $n = 5, i = 3$  is*

$$\mathcal{P}_5(I_{\{3\}}) = \text{conv}(\{(1, 1, 1, 0, 0), (0, 1, 1, 1, 0), (0, 0, 1, 1, 1)\}) = \text{conv}(\{\alpha_{1,3}, \alpha_{2,4}, \alpha_{3,5}\}).$$



**Figure 3.3:**  $\mathcal{P}_3(I_{\{1\}})$ .

**Example 27.** *The fixed interval vector polytope with  $n = 3, i = 1$  is*

$$\mathcal{P}_3(I_{\{1\}}) = \text{conv}(\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}) = \text{conv}(\{e_1, e_2, e_3\}).$$

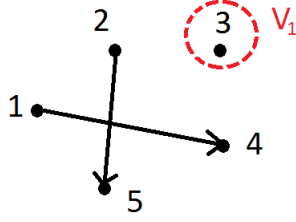
*This is a unit triangle in the affine subspace  $\{x + y + z = 1\}$  of  $\mathbb{R}^3$ .*

The last example is not an exception. In fact, the main theorem of this section shows us that that each fixed interval vector polytope is a unimodular simplex in its affine subspace. The proof requires some graph theory.

### 3.2.1 Flow Dimension Graphs

This entire construction is due to [Da]. Related to any interval vector polytope, there is associated a graph can be shown to tell us the affine-dimension of the polytope. Here I define the flow dimension graph, which will allows us to prove the main theorem of this section. For more information on flow dimension graphs and their construction and uses see [Da].

Denote  $e_{i,j} := e_i - e_j$  for  $i < j$ . We define the set of *elementary vectors* as



**Figure 3.4:** The flow dimension graph  $G_5(I_{\{3\}})$ .

containing all such  $e_{i,j}$ , each unit vector  $e_i$ , and the zero vector. Let  $T$  be the lower triangular  $(0, 1)$ -matrix, as in the proof of Theorem 2. We notice that  $T(e_i) = \alpha_{i,n}$  and  $T(e_{i,j}) = \alpha_{i,j-1}$ . So the image of an elementary vector is an interval vector. Since  $T$  is invertible, for any set of interval vectors  $I$ , there is a unique set  $\mathcal{E}$  of elementary vectors such that  $T(\mathcal{E}) = I$ , namely  $T^{-1}(I) = \mathcal{E}$ .

Thus for any interval vector polytope  $\mathcal{P}_n(I) \subset \mathbb{R}^n$ , we can construct the corresponding *flow-dimension graph*  $G_n(I) = (V, E)$  as follows. Let  $\mathcal{E} = T^{-1}(I)$ . We let the vertex set  $V = [n]$ , specify a subset  $V_1 = \{j \in V \mid e_j \in \mathcal{E}\}$ , and define the edge set  $E = \{(i, j) \mid e_{i,j} \in \mathcal{E}\}$ . Also we let  $k_0$  denote the number of connected components  $\mathcal{C}$  of the graph  $G$  (ignoring direction) so that  $\mathcal{C} \cap V_1 = \emptyset$ .

**Example 28.** Recall that the fixed interval-vector polytope with  $n = 5$ ,  $i = 3$  is

$$\mathcal{P}_5(I_{\{3\}}) = \text{conv}\{(1, 1, 1, 0, 0), (0, 1, 1, 1, 0), (0, 0, 1, 1, 1)\}$$

The corresponding flow dimension graph of  $G_5(I_{\{3\}}) = (V, E)$  has vertex set  $V = \{1, 2, 3, 4, 5\}$ . The corresponding elementary vectors are  $\mathcal{E} = \{e_{1,4}, e_{2,5}, e_3\}$ . Thus the edge set  $E = \{(1, 4), (2, 5)\}$  and the specified subset  $V_1 = \{3\} \subset V$ . The constant  $k_0$  representing the number of connected components of the graph not intersecting  $V_1$  is  $k_0 = 2$ .

**Theorem 3.** [Da] *If  $0 \in \text{aff}(I)$ , then the dimension of  $\mathcal{P}_n(I)$  is  $n - k_0$ . Else, if  $0 \notin \text{aff}(I)$  then the dimension of  $\mathcal{P}_n(I)$  is  $n - k_0 - 1$ .*

For  $\mathcal{P}_n(I_{\{i\}})$ , we have  $I = \{\alpha_{j,j+i-1} \mid j \leq n - i + 1\}$  which translates to the elementary vector set  $\mathcal{E} = \{e_{k,k+i} \mid k \leq n - i\} \cup \{e_{n-i+1}\}$ . We can define the corresponding flow-dimension graph  $G_n(I_{\{i\}}) = (V, E)$  where  $V = \{1, \dots, n\}$  and  $E = \{(k, k + i) \mid k \in [n - i]\}$  corresponding to each  $e_{i,j} \in \mathcal{E}$ . Also  $V_1 := \{n - i + 1\}$  corresponding to  $e_{n-i+1} \in \mathcal{E}$ .

**Lemma 3.** *Let  $a, b$  be nodes in the flow-dimension graph  $G_n(I_{\{i\}}) = (V, E)$ . Then  $a$  and  $b$  are connected iff  $a \equiv b \pmod{i}$ .*

*Proof.* Assume without loss of generality  $a \leq b$ . Suppose  $a$  and  $b$  are connected by the path  $q_0, \dots, q_s \in V$ . Therefore by definition of  $E$ , we have

$$\begin{aligned} q_0 &= a + i \\ q_1 &= q_0 + i = a + 2i \\ &\vdots \\ q_s &= q_{s-1} + i = a + (s + 1)i \\ b &= q_s + i = a + (s + 2)i \end{aligned}$$

Thus  $a \equiv b \pmod{i}$  by definition.

Now suppose that  $a \equiv b \pmod{i}$  where  $a \leq b$ , then there exists  $m \in \mathbb{N}$  such that

$$\begin{aligned} b &= a + mi \\ &= a + (m - 1)i + i. \end{aligned}$$

Since  $b$  and  $a + (m - 1)i$  differ by  $i$ , then by definition of  $E$ , there is an edge between these nodes. Call this edge  $(q_t, b) \in E$ . Similarly, we have

$$a + (m - 1)i = a + (m - 2)i + i \quad \Rightarrow (q_t, q_{t-1}) \in E$$

$$a + (m - 2)i = a + (m - 3)i + i \quad \Rightarrow (q_{t-1}, q_{t-2}) \in E$$

$\vdots$

$$a + 2i = (a + i) + i \quad \Rightarrow (q_1, q_0) \in E$$

$$a + i = a + i \quad \Rightarrow (q_0, a) \in E.$$

Hence  $q_0, q_1, \dots, q_t \in V$ , define a path from  $a$  to  $b$ , so  $a$  and  $b$  are connected.  $\square$

**Proposition 7.**  $\mathcal{P}_n(I_{\{i\}})$  is an  $(n - i)$ -dimensional simplex.

*Proof.* By Lemma 3 we know there are  $i$  connected components in the flow-dimension graph  $G_n(I_{\{i\}})$  and since  $V_1$  has only one element,  $k_0 = i - 1$ . Thus by Theorem 3 the dimension of  $\mathcal{P}_n(I_{\{i\}})$  is  $n - i$ . Since  $\mathcal{P}_n(I_{\{i\}})$  has  $n - i + 1$  vertices, it is an  $(n - i)$ -dimensional simplex.  $\square$

**Theorem 4.**  $\mathcal{P}_n(I_{\{i\}})$  is an  $(n - i)$ -dimensional unimodular simplex.

*Proof.* Consider the affine space where the sum over every  $i^{\text{th}}$  coordinate is 1,

$$A = \left\{ \mathbf{x} \in \mathbb{R}^n \mid \sum_{j \equiv k \pmod{i}} x_j = 1, \forall k \in [i] \right\}.$$

Since the vertices of  $\mathcal{P}_n(I_{\{i\}})$  have interval length  $i$ , they are in  $A$ . Thus  $\mathcal{P}_n(I_{\{i\}}) \subset A$ .

We want to show that the  $w_1, w_2, \dots, w_{n-i}$  of  $\mathcal{P}_n(I_{\{i\}})$  form a lattice basis for  $A$  where

$$\begin{aligned} w_1 &= \alpha_{1,i} - \alpha_{n-i+1,n} \\ w_2 &= \alpha_{2,i+1} - \alpha_{n-i+1,n} \\ &\vdots \\ w_{n-i} &= \alpha_{n-i,n-1} - \alpha_{n-i+1,n} \end{aligned}$$

We will do this by showing that any integer point  $p \in A$  can be expressed as a integral linear combination of the proposed lattice basis, that is, there exist integer coefficients  $Y_1, \dots, Y_{n-i}$  so that  $Y_1 w_1 + \dots + Y_{n-i} w_{n-i} + \alpha_{n-i+1,n} = p$ .

We first notice that  $p$  can be expressed as

$$\left( p_1, p_2, \dots, p_{n-i}, \sum_{\substack{j \leq n-i \\ j \equiv t-i+1 \pmod i}} (-p_j) + 1, \sum_{\substack{j \leq n-i \\ j \equiv t-i+2 \pmod i}} (-p_j) + 1, \dots, \sum_{\substack{j \leq n-i \\ j \equiv n \pmod i}} (-p_j) + 1 \right)$$

by solving for the last term in each of the equations defining  $A$ . Let

$$Y_t = \begin{cases} p_1 & t = 1 \\ p_t - p_{t-1} & 1 < t \leq i \\ p_t - Y_{t-i} & i < t \leq n - i \end{cases}$$

Then each  $Y_t$  is an integer. We claim that

$$Y_1 w_1 + \dots + Y_{n-i} w_{n-i} + \alpha_{n-i+1,n} = p.$$

Clearly the first coordinate is  $p_1$  since  $w_1$  is the only vector with an element in the first coordinate. Next consider the  $t^{\text{th}}$  coordinate of this linear combination for  $1 < t \leq i$ ,

by summing the coefficients of all the vectors who have a 1 in the  $t^{\text{th}}$  position:

$$Y_t + Y_{t-1} + Y_{t-2} + \cdots + Y_1 = p_t - p_{t-1} + p_{t-1} - p_{t-2} + \cdots + p_2 - p_1 + p_1 = p_t$$

We next consider the  $t^{\text{th}}$  coordinate of the combination for  $i < t \leq n - i$  by summing the coefficients of the vectors who have a 1 in the  $t^{\text{th}}$  position.

$$Y_t + Y_{t-1} + \cdots + Y_{t-i+1} = (p_t - Y_{t-1} - \cdots - Y_{t-i+1}) + Y_{t-1} + \cdots + Y_{t-i+1} = p_t$$

Finally, we consider the  $t^{\text{th}}$  coordinate of the combination for  $n - i < t \leq n$ , noticing that each coordinate from  $w_1$  to  $w_t$  has a  $-1$  in the  $(t - i)^{\text{th}}$  position and  $\alpha_{n-i+1, n}$  has a 1 in this position. Thus we get:

$$-(Y_1 + Y_2 + \cdots + Y_{t-i}) + 1.$$

Applying the two relations we have defined between coordinates, and calling  $\langle t \rangle$  the least residue of  $t \bmod i$ , we see:

$$\begin{aligned} -(Y_1 + Y_2 + \cdots + Y_{t-i}) + 1 &= -(Y_1 + Y_2 + \cdots + Y_{t-2i} + p_{t-i}) + 1 \\ &= -(Y_1 + Y_2 + \cdots + Y_{t-3i} + p_{t-2i} + p_{t-i}) + 1 \\ &= -\left( Y_1 + Y_2 + \cdots + Y_{\langle t \rangle} + \sum_{\substack{i < j \leq n-i \\ j \equiv t \pmod{i}}} p_j \right) + 1 \\ &= -\left( \sum_{\substack{j \leq n-i \\ j \equiv t \pmod{i}}} p_j \right) + 1. \end{aligned}$$

Thus  $p = Y_1 w_1 + Y_2 w_2 + \cdots + Y_{n-i} w_{n-i} + \alpha_{n-i+1, n}$  and so  $w_1, \dots, w_{n-i}$  form a lattice basis of  $A$ . So  $\mathcal{P}_n(I_{\{i\}})$  and is a unimodular simplex.  $\square$



**Corollary 5.**  $\text{vol } \mathcal{P}_n(I_{\{i\}}) = 1.$

To help with understanding this proof, I will walk through an example.

**Example 29.** Consider again  $P_5(I_{\{3\}})$  as in example 28. We saw that the number of connected components not intersecting  $V_1$  is  $k_0 = 2$ . We also know that  $0 \notin \text{aff}(I_{\{3\}})$  so that by theorem 3,  $\dim(\mathcal{P}_5(I_{\{3\}})) = 5 - 2 - 1 = 2$ . So  $\mathcal{P}_5(I_{\{3\}})$  is a 2 dimensional simplex. Since each vertex is, we also know that  $\mathcal{P}_5(I_{\{3\}})$  is a subset of the affine space

$$A = \left\{ \mathbf{x} \in \mathbb{R}^5 \mid x_1 + x_4 = 1, x_2 + x_5 = 1, x_3 = 1 \right\}.$$

But since this is 2 dimensional, we know that  $A$  is in fact the affine hull of  $\mathcal{P}_5(I_{\{3\}})$ . So to show that this polytope is a unimodular simplex, we must notice that its legs  $\{w_1, w_2\} = \{(1, 1, 1, 0, 0) - (0, 0, 1, 1, 1), (0, 1, 1, 1, 0) - (0, 0, 1, 1, 1)\}$  form a lattice basis for  $A$ .

Take any lattice point  $p \in A$ , it can be written as  $(x_1, x_2, 1, 1 - x_1, 1 - x_2)$ . If we let  $Y_1 = x_1$  and  $Y_2 = x_2 - x_1$ , then

$$\begin{aligned} Y_1 w_1 + Y_2 w_2 + \alpha_{3,5} &= x_1(1, 1, 0, -1, -1) + (x_2 - x_1)(0, 1, 0, 0, -1) + (0, 0, 1, 1, 1) \\ &= (x_1, x_1 + x_2 - x_1, 1, -x_1, -x_1 - x_2 + x_1 + 1) \\ &= (x_1, x_2, 1, 1 - x_1, 1 - x_2) = p. \end{aligned}$$

Since,  $\alpha_{3,5}$  is a lattice point of  $A$ ,  $p$  is an arbitrary lattice point of  $A$  and each  $Y_i$  is integral, then  $\{w_1, w_2\}$  is a lattice basis of  $A$  and  $P_5(I_{\{3\}})$  is unimodular.

# Chapter 4

## The Interval Pyramid

Given a dimension  $n$ , we define the *interval pyramid*  $\mathcal{P}_n(I_{\{1,n-1\}}) \subseteq \mathbb{R}^n$  to be the convex hull of all the standard unit vectors of  $\mathbb{R}^n$  along with the two interval vectors with interval length  $n - 1$ :  $\alpha_{1,n-1}$  and  $\alpha_{2,n}$

**Example 30.** For  $n = 3$  and  $n = 4$  and in general,

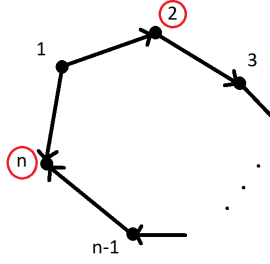
$$\mathcal{P}_3(I_{\{1,2\}}) = \text{conv}(\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (0, 1, 1)\}),$$

$$\mathcal{P}_4(I_{\{1,3\}}) = \text{conv}(\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (1, 1, 1, 0), (0, 1, 1, 1)\}),$$

$$\mathcal{P}_n(I_{\{1,n-1\}}) = \text{conv}(\{e_1, e_2, \dots, e_n, \alpha_{1,n-1}, \alpha_{2,n}\}).$$

We can use the flow dimension graph to show that  $\dim(P_n(I_{\{1,n-1\}})) = n$ , so that its affine space is all of  $\mathbb{R}^n$ .

**Proposition 8.** The dimension of  $\mathcal{P}_n(I_{\{1,n-1\}})$  is  $n$ .



**Figure 4.1:** The flow dimension graph of the interval pyramid:  $G(I_{\{1,n-1\}})$

*Proof.* For  $n \geq 3$ , the vertices of  $\mathcal{P}_n(I_{\{1,n-1\}})$  form the set

$$\mathcal{I} = \left\{ \begin{array}{l} e_1 = (1, 0, \dots, 0, 0) \\ e_2 = (0, 1, \dots, 0, 0) \\ \vdots \\ e_n = (0, 0, \dots, 0, 1) \\ \alpha_{1,n-1} = (1, 1, \dots, 1, 0) \\ \alpha_{2,n} = (0, 1, \dots, 1, 1) \end{array} \right\}.$$

We convert the interval vectors to the corresponding elementary vector set

$$\mathcal{E} = \{e_{1,2}, e_{2,3}, \dots, e_{n-1,n}, e_{1,n}, e_2, e_n\}.$$

From this we construct the flow-dimension graph  $G(\mathcal{P}_n(I_{\{1,n-1\}})) = (V, E)$  as seen in Figure 4.1, where  $V = [n]$  and

$$E = \{(k, k+1) | k \in [n-1]\} \cup \{(1, n)\}$$

corresponding to each  $e_{i,j}$  in  $\mathcal{E}$ . The subset of vertices  $V_1 = \{2, n\}$  (circled in Figure

4.1) corresponds to each  $e_i$  in  $\mathcal{E}$ . Since the underlying graph is connected, we know

$$k_0 = \#\{\text{connected components } C \text{ in } G(\mathcal{P}_n(I_{\{1,n-1\}})) \text{ such that } C \cap V_1 = \emptyset\} = 0.$$

Next we notice that

$$\frac{1}{n-2}e_1 + \frac{1}{n-2}e_2 + \cdots + \frac{1}{n-2}e_{n-1} - \frac{1}{n-2}\alpha_{1,n-1} = \mathbf{0}$$

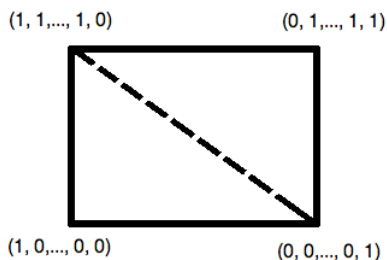
where the sum of the coefficients is

$$\frac{n-1}{n-2} - \frac{1}{n-1} = \frac{n-2}{n-2} = 1$$

So  $\mathbf{0} \in \text{aff}(\mathcal{I})$  and by Theorem 3,  $\dim(\mathcal{P}_n(I_{\{1,n-1\}})) = n - k_0 = n$ . □

## 4.1 $f$ -vector of the Interval Pyramid

Recall that the  $f$ -vector of a polytope tells us the number of faces a polytope has of each dimension. We will see that the  $f$ -vector of  $P_n(I_{\{1,n-1\}})$  with  $n \geq 3$ , is precisely the  $n^{\text{th}}$  row of the Pascal 3-triangle without 1's. The *Pascal 3-triangle* [Az] is an analogue of Pascal's Triangle, where the third row, instead of being 1 2 1, is replaced



**Figure 4.2:** The base of the interval pyramid.

with 1 3 1, and then the same addition pattern is followed as in Pascal's triangle.

$$\begin{array}{cccccc}
 n = 1: & & & & & 3 \\
 n = 2: & & & 4 & & 4 \\
 n = 3: & & & 5 & & 8 & & 5 \\
 n = 4: & & & 6 & & 13 & & 13 & & 6 \\
 n = 5: & & 7 & & 19 & & 26 & & 19 & & 7 \\
 n = 6: & 8 & & 26 & & 45 & & 45 & & 26 & & 8
 \end{array} \tag{4.1}$$

The proof of this correspondence requires a few preliminary results. First we define the face  $\mathcal{B} = \text{conv}(\{e_1, e_n, \alpha_{1,n-1}, \alpha_{2,n}\}) \subseteq \mathcal{P}_n(I_{\{1,n-1\}})$  as the *base* of the interval pyramid.

**Lemma 4.** *The base of the interval pyramid is 2 dimensional.*

*Proof.* We first consider  $\mathcal{A} = \text{conv}(e_n, \alpha_{1,n-1}, \alpha_{2,n})$ . The corresponding elementary vectors of the vertex set are  $\{e_{1,n}, e_2, e_n\}$ . So we build the flow-dimension graph as seen in Figure 4.1,  $G(\mathcal{A}) = (V, E)$  where  $V = [n]$ ,  $E = \{(1, n)\}$  corresponding to  $e_{1,n}$ . The subset  $V_1 = \{2, n\}$  (circled in Figure 2) corresponds to  $e_2$  and  $e_n$ . This

graph has  $n - 1$  connected components, two of which contain elements of  $V_1$  so that  $k_0 = n - 3$ .

If we let  $\lambda_1 e_n + \lambda_2 \alpha_{1,n-1} + \lambda_3 \alpha_{2,n} = \mathbf{0}$ , we first notice that  $\lambda_2 = 0$  since  $\alpha_{1,n-1}$  is the only vector with a nonzero first coordinate. But this implies that  $\lambda_1 = \lambda_3 = 0$ . Since the coefficients cannot sum to 1, we conclude that  $\mathbf{0} \notin \text{aff}(e_n, \alpha_{1,n-1}, \alpha_{2,n})$ .

So now by Theorem 3

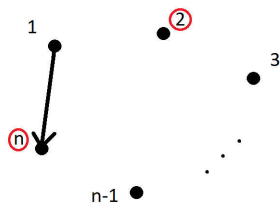
$$\dim(\text{conv}(e_n, \alpha_{1,n-1}, \alpha_{2,n})) = n - k_0 - 1 = n - (n - 3) - 1 = 2.$$

Finally  $e_1 = (1)\alpha_{1,n-1} + (-1)\alpha_{2,n} + (1)e_n$  is in the affine hull of  $\mathcal{A}$  and does not add a dimension. Thus we conclude that  $\dim(\mathcal{B}) = 2$ .  $\square$

**Corollary 6.** *Each  $e_i$  for  $2 \leq i \leq n - 1$  adds a dimension to  $\mathcal{P}_n(I_{\{1,n-1\}})$ , that is  $e_i \notin \text{aff } \mathcal{I}_{\{1,n-1\}} \setminus \{e_i\}$ .*

*Proof.* This follows from Theorem 4.1 and Lemma 4. Since the base of the interval pyramid  $\mathcal{B}$  has dimension 2 and  $\mathcal{P}_n(I_{\{1,n-1\}})$  has dimension  $n$ , then the  $n - 2$  remaining vertices must add the remaining  $n - 2$  dimensions. Clearly none can add more than one, so each must add precisely one dimension.  $\square$

**Lemma 5.** *The base of the interval pyramid,  $\mathcal{B}$  has  $f$ -vector  $(4, 4)$ .*



**Figure 4.3:**  $G(\mathcal{A})$ : The flow dimension graph of part of the base of the interval pyramid.

*Proof.* Since  $\mathcal{B}$  has dimension 2,  $f_1 = f_0$ . We know that  $\{e_n, \alpha_{1,n-1}, \alpha_{2,n}\}$  are three vertices of  $\mathcal{B}$  since they form a 2-dimensional object. If  $e_1 \in \text{conv}(e_n, \alpha_{1,n-1}, \alpha_{2,n})$  then

$$e_1 = \lambda_1 e_n + \lambda_2 \alpha_{1,n-1} + \lambda_3 \alpha_{2,n} \tag{4.2}$$

where the coefficients sum to 1. Since  $\alpha_{1,n-1}$  is the only vector with a nonzero coordinate in the first position, that implies  $\lambda_2 = 1$ . This in turn implies that  $\lambda_1 = \lambda_3 = 0$ , contradicting (4.2). So  $e_1 \notin \text{conv}(e_n, \alpha_{1,n-1}, \alpha_{2,n})$  and therefore forms a fourth vertex. Thus  $f_0 = 4 = f_1$  completing the proof.  $\square$

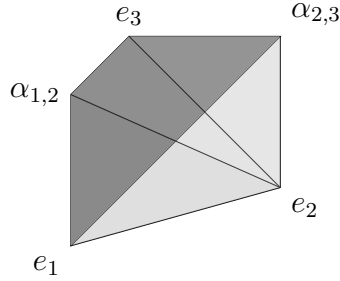
We can tie all this together with the following theorem. First we define (as in [Gr]) a *d-pyramid*  $P^d$  as the convex hull of the union of a  $(d-1)$ -dimensional polytope  $K^{d-1}$  (the *basis* of  $P^d$ ) and a point  $A \notin \text{aff}(K^{d-1})$  (the *apex* of  $P^d$ ).

**Theorem 5.** [Gr] *If  $P^d$  is a d-pyramid with  $(d-1)$ -dimensional basis  $K^{d-1}$  then*

$$\begin{aligned} f_0(P^d) &= f_0(K^{d-1}) + 1 \\ f_k(P^d) &= f_k(K^{d-1}) + f_{k-1}(K^{d-1}) \quad \text{for } 1 \leq k \leq d-2 \\ f_{d-1}(P^d) &= 1 + f_{d-2}(K^{d-1}). \end{aligned}$$

**Example 31.** Consider  $\mathcal{P}_3(I_{\{1,2\}})$ . The vertices of the base are  $\{e_1, e_3, \alpha_{1,2}, \alpha_{2,3}\}$ , which has  $f$ -vector  $(4, 4)$ . The vertex  $(0, 1, 0)$  serves as an apex over the base, and completes the pyramid, which now by Theorem 5 has  $f$ -vector  $(5, 8, 5)$ .

Since each vertex of the interval pyramid which is not a vertex of  $\mathcal{B}$  is affinely independent, we can imagine each one as the apex of a pyramid, and imagine the interval pyramid as being formed adding each apex one by one as a succession of pyramids over pyramidal bases all the way down to  $\mathcal{B}$ .



**Figure 4.4:** The 3 dimensional interval pyramid, with the base shaded in dark gray.

We notice that the rows of Pascal's 3-triangle act in the same manner as face numbers for pyramids, and we claim the face numbers for  $\mathcal{P}_n(I_{\{1,n-1\}})$  can be derived from Pascal's 3-triangle.

**Theorem 6.** *The  $f$ -vector for  $\mathcal{P}_n(I_{\{1,n-1\}})$  for  $n \geq 3$  is the  $n^{\text{th}}$  row of the Pascal 3-triangle.*

*Proof.* Let  $\mathcal{I} = \{e_1, e_2, \dots, e_n, \alpha_{1,n-1}, \alpha_{2,n}\}$  be the vertex set for  $\mathcal{P}_n(I_{\{1,n-1\}})$  with  $n \geq 3$ , and call  $\mathcal{R}_k = \text{conv}(\mathcal{I} \setminus \{e_k, e_{k+1}, \dots, e_{n-1}\})$  for  $1 \leq k < n$ . Then it is clear that  $\mathcal{P}_n(I_{\{1,n-1\}})$  is the convex hull of the union of the  $(n-1)$ -dimensional polytope  $\mathcal{R}_{n-1}$  and  $e_{n-1} \notin \text{aff}(\mathcal{R}_{n-1})$  (by Corollary 6), and thus is a pyramid and its face numbers can be computed as in Theorem 5 from the face numbers of  $\mathcal{R}_{n-1}$ .

Notice next that  $\mathcal{R}_{n-1}$  is the convex hull of the union of the  $(n-2)$ -dimensional polytope  $\mathcal{R}_{n-2}$  and  $e_{n-2} \notin \text{aff}(\mathcal{R}_{n-2})$  (again by Corollary 6), so we can compute the face numbers of  $\mathcal{R}_{n-1}$  from those of  $\mathcal{R}_{n-2}$  as in Theorem 5.

We can continue this process until we get that  $\mathcal{R}_3$  is the convex hull of  $\mathcal{R}_2$  and  $e_2 \notin \text{aff}(\mathcal{R}_2)$ . But we notice that  $\mathcal{R}_2 = \mathcal{B}$  is the base of the interval pyramid, so by Lemma 5,  $f_0(\mathcal{R}_2) = f_1(\mathcal{R}_2) = 4$ . From here we can build using  $f$ -vectors of  $\mathcal{P}_n(I_{\{1,n-1\}})$  from Theorem 5 which are exactly those of the Pascal 3-triangle. We do this  $n-1$  times to reach  $\mathcal{P}_n(I_{\{1,n-1\}})$ , and since  $(4,4)$  is the second row of the



triangle, then the  $f$ -vector of  $\mathcal{P}_n(I_{\{1,n-1\}})$  is the  $n^{\text{th}}$  row of the Pascal 3-triangle, as desired.  $\square$

We can rewrite (4.1) as

$$\begin{array}{cccccc}
& & & & & 2 + \mathbf{1} \\
& & & & & & 3 + \mathbf{1} & & & 3 + \mathbf{1} \\
& & & & & 4 + \mathbf{1} & & 6 + \mathbf{2} & & 4 + \mathbf{1} \\
& & & 5 + \mathbf{1} & & 10 + \mathbf{3} & & 10 + \mathbf{3} & & 5 + \mathbf{1} \\
& 6 + \mathbf{1} & & 15 + \mathbf{4} & & 20 + \mathbf{6} & & 15 + \mathbf{4} & & 6 + \mathbf{1} \\
7 + \mathbf{1} & & 21 + \mathbf{5} & & 35 + \mathbf{10} & & 35 + \mathbf{10} & & 21 + \mathbf{5} & & 7 + \mathbf{1}
\end{array} \tag{4.3}$$

which is Pascal's triangle added to a shifted Pascal's triangle. Thus we can derive a formula for the number of  $k$ -faces for  $\mathcal{P}_n(I_{\{1,n-1\}})$  in terms of binomial coefficients.

**Corollary 7.** For  $n \geq 3$ ,  $f_k(\mathcal{P}_n(I_{\{1,n-1\}})) = \binom{n-1}{k} + \binom{n+1}{k+1}$ .

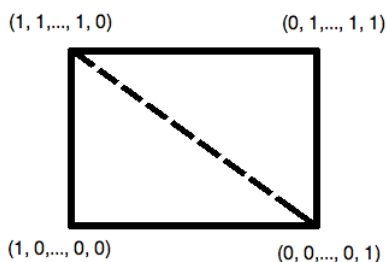
## 4.2 Volume of the Interval Pyramid

The base  $\mathcal{B}$  of the interval pyramid can be easily triangulated into

$$\Delta_1 = \text{conv}(e_1, e_n, \alpha_{1,n-1}) \text{ and } \Delta_2 = \text{conv}(e_n, \alpha_{1,n-1}, \alpha_{2,n}).$$

Since the remaining vertices are affinely independent and triangulations of a base extend to a pyramid, this extends to a triangulation of  $\mathcal{P}_n(I_{1,n-1})$  into 2  $n$ -simplexes

$$S_1 = \text{conv}(e_1, e_2, \dots, e_{n-1}, e_n, \alpha_{1,n-1}) \text{ and } S_2 = \text{conv}(e_2, \dots, e_{n-1}, e_n, \alpha_{1,n-1}, \alpha_{2,n}).$$



**Figure 4.5:** Triangulation of the base of the interval pyramid

We know these are full dimensional, so we can recall from Proposition 2 that the volume of a simplex is easily computed using the Cayley Menger determinant. Recall that for  $\mathcal{P} \in \mathbb{R}^n$ , a full dimensional  $n$ - simplex with vertex set  $\{v_0, \dots, v_n\}$ ,

$$\text{vol}(\mathcal{P}) = \det(v_1 - v_0, v_2 - v_0, \dots, v_n - v_0).$$

So this triangulation allows us to easily calculate the volume of the interval pyramid.

**Lemma 6.** *The determinant of the  $n \times n$ -matrix*

$$\begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ & & & \ddots & \\ 1 & \cdots & 1 & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}$$

is  $(-1)^{n-1}(n-1)$ .

*Proof.* Let  $A_n$  be the  $n \times n$  matrix whose diagonal entries are 0, and all entries off the diagonal are 1. E.g.,

$$A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and so  $\det(A_2) = -1$ . Assume  $\det(A_k) = (-1)^{k-1}(k-1)$ .  $A_{k+1}$  is the  $(k+1) \times (k+1)$ -matrix of the form

$$A_{k+1} = \begin{bmatrix} 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ & & \ddots & & \\ 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}.$$

Subtracting the second row from the first, which does not change the value of the determinant, will give us the matrix

$$\begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 1 \\ & & \ddots & & \\ 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}.$$

Now the determinant of  $A_{k+1}$  is the sum of two determinants by cofactor expansion. Specifically it is  $(-1) \det(A_k)$  minus the determinant of the matrix obtained by taking out the first row and second column. We know that  $(-1) \det(A_k) = (-1)^k(k-1)$  by the inductive hypothesis. So what we have left to compute is the determinant of the

$(k \times k)$ -matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ & & \ddots & & \\ 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}.$$

We will subtract the first row from each of the rows below it, also not changing the determinant, to give us the upper triangular matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & 1 \\ 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 \\ & & & \ddots & & \\ 0 & 0 & \cdots & 0 & -1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & -1 \end{bmatrix}$$

whose determinant is  $(-1)^{k-1}$ . Furthermore,

$$\begin{aligned} \det(A_{k+1}) &= (-1) \det(A_k) - (-1)^{k-1} \\ &= (-1)^k (k-1) + (-1)^k \\ &= (-1)^k k. \end{aligned}$$

Therefore, by induction,  $\det(A_n) = (-1)^{n-1} (n-1)$ , for all  $n \in \mathbb{Z}_{\geq 2}$ . □

**Theorem 7.**  $\text{vol}(\mathcal{P}_n(I_{\{1, n-1\}})) = 2(n-2)$  for  $n \geq 3$

*Proof.* In order to calculate the volume of  $\mathcal{P}_n(I_{\{1, n-1\}})$  we will first triangulate the

2-dimensional base of the pyramid from Lemma 4

$$\Delta_1 = \text{conv}(e_1, e_n, \alpha_{1,n-1}) \text{ and } \Delta_2 = \text{conv}(e_n, \alpha_{1,n-1}, \alpha_{2,n}).$$

Let  $x$  be a point in the base, then for some  $\lambda_i \geq 0$ , where  $\sum_{i=1}^4 \lambda_i = 1$ ,

$$\begin{aligned} x &= \lambda_1 e_1 + \lambda_2 e_n + \lambda_3 \alpha_{1,n-1} + \lambda_4 \alpha_{2,n} \\ &= (\lambda_1 + \lambda_3, \lambda_3 + \lambda_4, \dots, \lambda_3 + \lambda_4, \lambda_2 + \lambda_4) \\ &= (\lambda_1 - \lambda_4) e_1 + (\lambda_2 + \lambda_4) e_n + (\lambda_3 + \lambda_4) \alpha_{1,n-1} \\ &= (\lambda_1 + \lambda_2) e_n + (\lambda_1 + \lambda_3) \alpha_{1,n-1} + (\lambda_4 - \lambda_1) \alpha_{2,n}. \end{aligned}$$

So  $x$  is a point in  $\Delta_1$  if  $\lambda_1 \geq \lambda_4$  and  $x$  is a point in  $\Delta_2$  if  $\lambda_4 \geq \lambda_1$ . Thus  $\Delta_1$  and  $\Delta_2$  is a triangulation of the 2-dimensional base of the pyramid.

By Corollary 6, each  $e_2, \dots, e_{n-1}$  adds a dimension so that the convex hull of these points and  $\Delta_1$  is an  $n$ -dimensional simplex. The same can be said of  $\Delta_2$ . Call these simplices  $S_1$  and  $S_2$  respectively. Thus  $S_1$  and  $S_2$  triangulate  $\mathcal{P}_n(I_{\{1,n-1\}})$ . Therefore the sum of their volumes is equal to the volume of  $\mathcal{P}_n(I_{\{1,n-1\}})$ . In order to calculate the volume of  $S_1$  and  $S_2$ , we will use the Cayley Menger determinant [Da] once again. Consider  $S_1$ , whose volume is the determinant of the matrix

$$\begin{bmatrix} e_1 - \alpha_{1,n-1} & e_2 - \alpha_{1,n-1} & \dots & e_n - \alpha_{1,n-1} \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 & \dots & -1 & -1 \\ -1 & 0 & -1 & \dots & -1 & -1 \\ -1 & -1 & 0 & -1 & \dots & -1 \\ & & & \ddots & & \\ -1 & -1 & \dots & -1 & 0 & -1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

Cofactor expansion on the last row will leave us with the determinant, up to a sign, of the  $(n - 1) \times (n - 1)$  matrix

$$\begin{bmatrix} 0 & -1 & -1 & \cdots & -1 \\ -1 & 0 & -1 & \cdots & -1 \\ & & \ddots & & \\ -1 & \cdots & -1 & 0 & -1 \\ -1 & -1 & \cdots & -1 & 0 \end{bmatrix}, \quad (4.4)$$

which when ignoring sign by Lemma 6 is  $n - 2$ . Therefore the volume of  $S_1$  is  $n - 2$ .

Now consider the Cayley Menger determinant of  $S_2$ , the determinant of

$$\begin{bmatrix} \alpha_{1,n-1} - \alpha_{2,n} & e_2 - \alpha_{2,n} & e_3 - \alpha_{2,n} & \cdots & e_n - \alpha_{2,n} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & -1 & \cdots & -1 \\ 0 & -1 & 0 & -1 & \cdots & -1 \\ & & & \ddots & & \\ 0 & -1 & -1 & \cdots & 0 & -1 \\ -1 & -1 & -1 & \cdots & -1 & 0 \end{bmatrix}.$$

By cofactor expansion on the first row we are left with the positive determinant of the matrix (4.4) which is  $n - 2$ . Therefore the volume of  $S_2$  is  $n - 2$  and so the volume of  $\mathcal{P}_n(I_{\{1,n-1\}})$  is  $2(n - 2)$ , as desired.  $\square$

### 4.3 Duality of the Interval Pyramid

Since the interval pyramid has a symmetric  $f$ -vector, it is natural to wonder about its duality. That is, if we replace each  $k$  face with an  $n - k$  face (take the dual of the

polytope), are you left with an identical polytope? How does the volume and face lattice structure change? Clearly the  $f$ -vector will remain the same, but I wonder whether the interval pyramid is actually self-dual.

Recall that in Section 2.3.3, we offered the following construction to define the dual of a polytope  $\mathcal{P} \subset \mathbb{R}^n$  for the cases where  $0 \in \text{int}(\mathcal{P})$ . [Zi]

$$\mathcal{P}^\Delta := \{y \in \mathbb{R}^n : y \cdot x \leq 1 \text{ for all } x \in \mathcal{P}\},$$

where  $y \cdot x = y_1x_1 + \dots + y_nx_n$  is the dot product. But we know that  $0 \notin \text{int}(\mathcal{P}_n(I_{\{1, n-1\}}))$  for any interval pyramid. Still, we noted in the previous section that we can affinely translate a polytope so that 0 is in the interior without losing any information about the face structure or volume.

Let us take several examples of translating the interval pyramid so that it contains zero and taking the dual.

**Example 32.** Consider  $\mathcal{P}_3(I_{\{1,2\}}) = \text{conv}((1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (0, 1, 1))$ .

Call the vertex set  $V$ . Let  $v = (\frac{1}{2}, \frac{1}{2}, \frac{1}{4})$  The affine translation

$$(\mathcal{P}_3(I_{\{1,2\}}))_v = \text{conv}\left(V - \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{4}\right)\right)$$

leaves the structure of the polytope intact and has zero in its interior as we will show.

First let's call the translated vertices of  $(\mathcal{P}_3(I_{\{1,2\}}))_v$  (in order)  $v_1, \dots, v_5$ . Then notice that

$$v_1 + v_2 + v_3 + v_4 = \begin{pmatrix} 1/2 \\ -1/2 \\ -1/4 \end{pmatrix} + \begin{pmatrix} -1/2 \\ 1/2 \\ -1/4 \end{pmatrix} + \begin{pmatrix} -1/2 \\ -1/2 \\ 1/4 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 1/2 \\ -1/4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus if we let  $\lambda_1 = \dots = \lambda_4 = 1/4$  and  $\lambda_5 = 0$ . Then

$$\lambda_1 v_1 + \dots + \lambda_5 v_5 = \frac{1}{4}(v_1 + \dots + v_4) = 0$$

With  $\lambda_1 + \dots + \lambda_5 = 1$  and each  $\lambda_i \geq 0$ .

Applying to dual construction to this translated polytope gives us a half space description for  $(\mathcal{P}_3(I_{\{1,2\}}))_v^\Delta$  given by the following inequalities:

$$\begin{aligned} \frac{1}{2}x_1 - \frac{1}{2}x_2 - \frac{1}{4}x_3 &\leq 1 \\ -\frac{1}{2}x_1 + \frac{1}{2}x_2 - \frac{1}{4}x_3 &\leq 1 \\ -\frac{1}{2}x_1 - \frac{1}{2}x_2 + \frac{3}{4}x_3 &\leq 1 \\ \frac{1}{2}x_1 + \frac{1}{2}x_2 - \frac{1}{4}x_3 &\leq 1 \\ -\frac{1}{2}x_1 + \frac{1}{2}x_2 + \frac{3}{4}x_3 &\leq 1. \end{aligned}$$

**Example 33.** Consider  $\mathcal{P}_4(I_{\{1,3\}}) = \text{conv}(\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (1, 1, 1, 0), (0, 1, 1, 1)\})$ . Call the vertex set  $V$ . Letting  $u = (\frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{1}{5})$ , we can translate this polytope so that  $0$  is an interior point as follows:

$$(\mathcal{P}_4(I_{\{1,3\}}))_u = \text{conv}\left(V - \left(\frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{1}{5}\right)\right).$$

Now we can apply the dual construction and have  $((\mathcal{P}_4(I_{\{1,3\}}))_u)^\Delta$  given by the following



inequalities:

$$\begin{aligned}
\frac{3}{5}x_1 - \frac{2}{5}x_2 - \frac{2}{5}x_3 - \frac{1}{5}x_4 &\leq 1 \\
-\frac{2}{5}x_1 + \frac{3}{5}x_2 - \frac{2}{5}x_3 - \frac{1}{5}x_4 &\leq 1 \\
-\frac{2}{5}x_1 - \frac{2}{5}x_2 + \frac{3}{5}x_3 - \frac{1}{5}x_4 &\leq 1 \\
-\frac{2}{5}x_1 - \frac{2}{5}x_2 - \frac{2}{5}x_3 + \frac{4}{5}x_4 &\leq 1 \\
\frac{3}{5}x_1 + \frac{3}{5}x_2 + \frac{3}{5}x_3 - \frac{1}{5}x_4 &\leq 1 \\
-\frac{2}{5}x_1 + \frac{3}{5}x_2 + \frac{3}{5}x_3 + \frac{4}{5}x_4 &\leq 1.
\end{aligned}$$

We want to prove that 0 is an interior point of the translated interval pyramid.

We first recall lemma 1[Zi].

**Lemma 7.** [Zi] *Let  $\mathcal{P}$  be a polytope in  $\mathbb{R}^n$ . If  $p \in \mathcal{P}$  can be represented as  $p = \frac{1}{n+1} \sum_{i=0}^n x_i$  for  $n+1$  affinely independent points  $x_0, \dots, x_n \in \mathcal{P}$ , then  $p$  is an interior point of  $\mathcal{P}$ .*

**Proposition 9.** *Let  $\mathcal{P}_n(I_{\{1, n-1\}})$  be the  $n$  dimensional interval pyramid. Let  $u = (\frac{2}{n+1}, \frac{2}{n+1}, \dots, \frac{2}{n+1}, \frac{1}{n+1})$ . Then 0 is an interior point of  $(\mathcal{P}_n(I_{\{1, n-1\}}))_u$ . This affine translation doesn't disturb any combinatorial structure (i.e. it is a purely linear translation leaving all structure of the polytope intact, essentially just repositioning the origin).*

*Proof.* First we show that  $0 \in (\mathcal{P}_n(I_{\{1, n-1\}}))_u$ . The vertex set of  $(\mathcal{P}_n(I_{\{1, n-1\}}))_u$  is

$$\{v_1, \dots, v_{n+2}\} = \{e_1 - u, e_2 - u, \dots, e_n - u, \alpha_{1, n-1} - u, \alpha_{2, n} - u\}.$$

Recall that  $\alpha_{1,n-1} = e_1 + \dots + e_{n-1}$  and notice that:

$$\begin{aligned}
v_1 + \dots + v_{n+1} &= (e_1 - u) + \dots + (e_n - u) + (\alpha_{1,n-1} - u) \\
&= e_1 + \dots + e_n + (e_1 + \dots + e_{n-1}) - (n+1)u \\
&= 2e_1 + \dots + 2e_{n-1} + e_n - (2, \dots, 2, 1) = 0
\end{aligned}$$

So if we let  $\lambda_1 = \dots = \lambda_{n+1} = \frac{1}{n+1}$  and  $\lambda_{n+2} = 0$ , then

$$\lambda_1 v_1 + \dots + \lambda_{n+2} v_{n+2} = \frac{1}{n+1} (v_1 + \dots + v_{n+1}) = 0$$

Notice that  $v_1, \dots, v_{n+1}$  are  $n+1$  affinely independent points. This is easier to show using the untranslated vertices  $e_1, \dots, e_n, \alpha_{1,n-1}$  (which is equivalent). Clearly the unit vectors are each affinely independent, and  $\text{aff}(e_1, \dots, e_n) = \{x \in \mathbb{R}^n : \sum_i x_i = 1\} \not\ni \alpha_{1,n-1}$ . So since  $v_1, \dots, v_{n+1}$  are  $n+1$  affinely independent points in  $\mathcal{P}_n(I_{\{1,n-1\}}) - u$ , we can refer to the previous lemma to notice that since  $0 = \frac{1}{n+1} \sum_{i=1}^{n+1} v_i$  then 0 is in fact an interior point of  $(\mathcal{P}_n(I_{\{1,n-1\}}))_u$ .  $\square$

We recall Proposition 6 to take the dual construction and calculate the half space description of the interval pyramid in general (albeit translated).

**Proposition 10.** *Let  $\mathcal{P} \subseteq \mathbb{R}^n$  be a polytope with 0 on its interior, and vertex set  $\{v_1, \dots, v_m\}$ . If  $c \in \mathbb{R}^n$ , then  $c \in \mathcal{P}^\Delta$  if and only if  $c \cdot v_i \leq 1$  for  $i = 1, \dots, m$ .*

This is enough to give us the facet description we need.

**Corollary 8.** *Let  $u = (\frac{2}{n+1}, \frac{2}{n+1}, \dots, \frac{2}{n+1}, \frac{1}{n+1})$  and assume that  $\mathcal{P}_n(I_{\{1,n-1\}})_u$  has vertices  $\{v_1, \dots, v_{n+2}\}$ . Then we define the dual by:*

$$(\mathcal{P}_n(I_{\{1,n-1\}}))_u^\Delta = \{c \in \mathbb{R}^n : c \cdot v_i \leq 1 \text{ for } i = 1, \dots, n+2\}$$

Thus,  $(\mathcal{P}_n(I_{\{1,n-1\}}))_u^\Delta$  has the following inequality description.

$$\begin{aligned} \left( \sum_{i \in [n-1], i \neq j} \frac{-2}{n+1} x_i \right) + \left( 1 - \frac{2}{n+1} \right) x_j - \frac{1}{n+1} x_n &\leq 1 \text{ for all } j \in [n-1] \\ \left( \sum_{i=1}^{n-1} \left( -\frac{2}{n+1} x_i \right) \right) + \frac{n}{n+1} x_n &\leq 1 \\ \left( \sum_{i=1}^{n-1} \left( 1 - \frac{2}{n+1} \right) x_i \right) - \frac{1}{n+1} x_n &\leq 1 \\ \frac{-2}{n+1} x_1 + \left( \sum_{i=2}^{n-1} \left( 1 - \frac{2}{n+1} \right) \right) + \frac{n}{n+1} x_n &\leq 1. \end{aligned}$$

*Proof.* Because we know that the  $f$ -vector of the interval pyramid is symmetric, we know that there are exactly  $n+2$  facets of the dual of the interval pyramid. Because of the Proposition 10, we know that a point is in the dual of the interval pyramid if and only if it satisfies the  $n+2$  facet equations given, so these facets are sufficient to completely describe the dual of the interval pyramid. But because we know we need at least  $n+2$  facets, we know as well that they are necessary. Thus this is a complete facet description of  $(\mathcal{P}_n(I_{\{1,n-1\}}))_u^\Delta$ .  $\square$

Recall that two polytopes are called *combinatorially isomorphic* if their face lattices are isomorphic as posets. If a polytope is combinatorially isomorphic to its dual then we call the polytope self dual. Let  $u$  be as defined in the previous corollary, to prove that the interval pyramid is self dual we must prove that

$$L(((\mathcal{P}_n(I_{\{1,n-1\}}))_u^\Delta) \cong L(\mathcal{P}_n(I_{\{1,n-1\}})).$$

But by proposition 2 we know that

$$L((\mathcal{P}_n(I_{\{1,n-1\}}))_u^\Delta) \cong L((\mathcal{P}_n(I_{\{1,n-1\}}))^{op}).$$

And because the affine translation preserved all of the combinatorial structure of the polytope we know that:

$$L((\mathcal{P}_n(I_{\{1,n-1\}})_u)^{op}) \cong L((\mathcal{P}_n(I_{\{1,n-1\}})))^{op}.$$

So in fact, to show that the interval pyramid is self dual, we need only to show that:

$$(L(\mathcal{P}_n(I_{\{1,n-1\}})))^{op} \cong L(\mathcal{P}_n(I_{\{1,n-1\}}))$$

To fully understand the combinatorial self-duality of the interval pyramid, we must fully describe its face lattice in general.

# Chapter 5

## Open Questions

First we note that we have not yet answered the question on the self-duality of the Interval Pyramid. To do so, we must get a general look at its face lattice poset and see if we can prove an isomorphism between that poset and its opposite. Computing examples in Sage and Polymake for the first 10 Interval Pyramids leads me to this first conjecture.

**Conjecture 1.** *Let  $\mathcal{P}_n(I_{\{1,n-1\}}) \subset \mathbb{R}^n$  be the interval pyramid.*

$$(L(\mathcal{P}_n(I_{\{1,n-1\}})))^{op} \cong L(\mathcal{P}_n(I_{\{1,n-1\}}))$$

*That is, the interval pyramid is self dual.*

As the nature of this project was cataloguing an interesting class of combinatorial polytopes, and we only touched on a few initially, there is plenty of interesting work to be done further investigating polytopes with different sets of interval vectors as vertices. Since all of the collections of interval vectors we've considered thus far have yielded very interesting combinatorial properties when considered as vertices of lattice polytopes, there is a good chance that there are many interesting interval

vector polytopes left to study. Some possibilities are  $\mathcal{P}_n(I_{\geq i})$  or  $\mathcal{P}_n(I_{\leq i})$ , where the interval vectors all have length greater than or less than some fixed constant. Here is one more interval vector polytope we have begun doing work on.

## 5.1 Generalized Interval Pyramid

We can make a generalization of the previous polytope by defining the *generalized interval pyramid*  $\mathcal{P}_n(I_{\{1, n-i\}})$  to be the convex hull of all the standard unit vectors in  $\mathbb{R}^n$  and all the interval vectors with interval length  $n - i$ . We restrict ourselves to the cases where  $n \geq 2$  and  $i \leq \frac{n}{2}$ .

**Example 34.** For  $n = 6$  and  $i = 2$ ,

$$\mathcal{P}_6(I_{\{6,2\}}) = \text{conv}(e_1, e_2, e_3, e_4, e_5, e_6, \alpha_{1,3}, \alpha_{2,4}, \alpha_{3,5}, \alpha_{4,6}).$$

We are able to generalize many of the results from the interval pyramid. The important similarity is that the generalized interval pyramid is an  $n$  dimensional polytope, which is formed as a series of pyramids over a  $2i$  dimensional base. I hope to focus on triangulating this base in general. Repeated computations in of examples in polymake [Ga] have led to the following conjecture.

**Conjecture 2.**  $\text{vol}(\mathcal{P}_n(I_{\{1, n-i\}})) = 2^i(n - (i + 1))$

In all the examples I considered, the  $f$ -vector was a combination of pascals triangle rows, and so the face numbers should all be easily expressible in terms of binomial coefficients. This is an area for further study.

# Bibliography

- [Az] Azarian, M. *Pascal 3-Triangle*, OEISA028263.
- [Dm] Dmharvey *Catalan number 4x4 grid example*  
[http://upload.wikimedia.org/wikipedia/commons/f/f4/Catalan\\_number\\_4x4\\_grid\\_example.svg](http://upload.wikimedia.org/wikipedia/commons/f/f4/Catalan_number_4x4_grid_example.svg).
- [Ep] Eppstein D. *The face lattice of a square pyramid*,  
[https://upload.wikimedia.org/wikipedia/commons/thumb/4/4c/Pyramid\\_face\\_lattice.svg/500px-Pyramid\\_face\\_lattice.svg.png](https://upload.wikimedia.org/wikipedia/commons/thumb/4/4c/Pyramid_face_lattice.svg/500px-Pyramid_face_lattice.svg.png).
- [Ks] KSmrq *Hasse Diagram of  $\{x, y, z\}$* ,  
[http://upload.wikimedia.org/wikipedia/commons/e/ea/Hasse\\_diagram\\_of\\_powerset\\_of\\_3.svg](http://upload.wikimedia.org/wikipedia/commons/e/ea/Hasse_diagram_of_powerset_of_3.svg).
- [BDDPR] Beck, M., De Silva, J., Dorfsman-Hopkins, G., Pruitt, J., Ruiz, A. *The Combinatorics of Interval Vector Polytopes*,  
<http://arxiv.org/abs/1211.2039>, 2012.
- [Ga] Gawrilow, E., Joswig, M. *polymake: a framework for analyzing convex polytopes*, Polytopes—combinatorics and computation (Oberwolfach, 1997).
- [Be] Beck, M., Robins, S. *Computing the continuous discretely, Integer-point enumeration in polyhedra*, Springer, Undergraduate Texts in Mathematics, 2007.
- [Da] Dahl, G. *Polytopes related to interval vectors and incidence matrices*, Linear Algebra and its Applications, 435 (2011) no. 11, 2955-2960.
- [Po] Postnikov, A. *Permutohedra, associahedra, and beyond* Int. Math. Res. Not., 2009 no. 6, 1026-1106
- [Gr] Grünbaum, B. *Convex polytopes*, Springer, Graduate Texts in Mathematics, 2003.
- [Da] D’Andrea, K., Sombra, M. *The Cayley-Menger determinant is irreducible for  $n \geq 3$* , Sibirsk. Mat. Zh., 46 (2005) no. 1, 90-97.

- [Zi] Ziegler, G. *Lectures on polytopes*, Springer, Graduate Texts in Mathematics, 1995.
- [St] Stanley, R. *Enumerative combinatorics. Volume 1*, Second Ed, Cambridge University Press, 2012