Dynamics of Magnetic Flows on Nilmanifolds

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Abstract

In this thesis, we study the dynamics of magnetic flows on compact nilmanifolds. Magnetic flows are generalizations of geodesic flows. They model the motion of a particle of unit mass and unit charge in a smooth manifold M in the presence of a magnetic field. As such, their dynamical properties are influenced by both the underlying Riemannian geometry and the closed 2-form on M which plays the role of the magnetic field. At the same time, nilmanifolds are a rich and varied class of examples, as well as a source of conjectures and counterexamples in Riemannian geometry. More precisely, we consider nilmanifolds of the form $M = \Gamma\backslash G$, where G is a simply connected 2-step nilpotent Lie group and $\Gamma < G$ is a cocompact discrete subgroup. The manifold M is endowed with a Riemannian metric g and closed 2-form, or magnetic field, σ, each of which pulls back to a left-invariant tensor field on G.

First, we focus on the case when G is the $2n + 1$ dimensional Heisenberg group, and σ is exact. We calculate the Mañé critical value and the lengths of closed magnetic geodesics in nontrivial free homotopy classes. Next we consider the topological entropy of magnetic flows on arbitrary 2-step compact nilmanifolds. When σ represents a rational cohomology class and its restriction to $\mathfrak{g} = T_eG$ vanishes on the derived algebra, we prove that the associated magnetic flow has zero topological entropy on a dense set of energy levels. In particular, this is the case when σ represents a rational cohomology class and is exact. Lastly, we provide an example of a magnetic field on a 2-step compact nilmanifold that has positive topological entropy for arbitrarily high energy levels. The salient difference in this case is that σ is not exact. We discuss the relationship to Mañé's critical value. The main tool is a symplectic reduction of the cotangent bundle of a nilmanifold of one more dimension than M.