Numerically explicit estimates for character sums

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Character sums make their appearance in many number theory problems: showing that there are infinitely many primes in any coprime arithmetic progression, estimating the least quadratic non-residue, bounding the least primitive root, finding the size of the least inert prime in a real quadratic field, etc. In this thesis, we find numerically explicit estimates for character sums and give applications to some of these questions.

Granville, Mollin and Williams proved that the least inert prime q for a real quadratic field of discriminant D such that $D > 3705$, satisfies $q \leq \sqrt{D}/2$. Using a smoothed version of the Pólya–Vinogradov inequality (an explicit bound on character sums) and explicit estimates on the sum of primes, we improve the bound on q to $D^{0.45}$ for $D > 1596$.

Let χ be a non-principal Dirichlet character $\bmod p$ for a prime p. Using combinatorial methods, we improve an inequality of Burgess for the double sum

$$\sum_{m=1}^{p} \left| \sum_{l=0}^{h-1} \chi(m+l) \right|^{2w}.$$

Using this inequality, we prove that for a prime p with $k \mid p-1$, the least k-th power non-residue $\bmod p$ is smaller than $0.9p^{1/4}\log p$ unless $k = 2$ and $p \equiv 3 \pmod 4$, in which case, the least k-th power non-residue is smaller than $1.1p^{1/4}\log p$. This improves a result of Norton which has the coefficients 3.9 and 4.7 in the two cases, respectively. We also prove that the length H of the longest interval on which χ is constant is smaller than $3.64p^{1/4}\log p$ and if $p \geq 2.5 \cdot 10^9$, then $H \leq 1.55p^{1/4}\log p$. This improves a result of McGown which had for $p \geq 5 \cdot 10^{18}$ that $H \leq 7.06p^{1/4}\log p$, and for $p \geq 5 \cdot 10^{55}$ that $H \leq 7p^{1/4}\log p$.

The purpose of this thesis is to work out the best explicit estimates we can and to have them as tools for other mathematicians.