Coloring Posets and Reverse Mathematics

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Abstract

We study two themes from Reverse Mathematics. The first theme involves a generalization of the infinite version of Ramsey's theorem to arbitrary partial orderings. We say that a partial ordering $\mathbb{P}$ has the (n, k)-Ramsey property, and write $RT^n_k(\mathbb{P})$, if for every k-coloring of the n-element chains of $\mathbb{P}$ there is a homogeneous copy of $\mathbb{P}$.

When $\mathbb{P}$ is either a linear ordering or a tree, and $n \geq 3$, the statement $(\forall k \geq 1)RT^n_k(\mathbb{P})$ is well understood from the point of view of Reverse Mathematics. We investigate $RT^n_k(\mathbb{P})$ for some partial orderings which are not trees. We show that if $\mathbb{P}$ is either the binary tree with multiplicities or an amenable partial ordering, and if $n \geq 3$, then the statement $(\forall k \geq 1)RT^n_k(\mathbb{P})$ is equivalent to ACA_0 over RCA_0. We also classify which suborderings of the binary tree with multiplicities have the Ramsey property. Finally, we study the $(1, k)$-Ramsey property for the finite (ordinal) powers of ω. For these orderings it makes sense to consider a first-order definition of "an isomorphic copy of ω^n" and the corresponding version of $\forall k RT^1_k(\omega^n)$, which we denote by $\mathsf{Elem\text{-}Indec}^n$. We place a lower bound on the complexity of $\mathsf{Elem\text{-}Indec}^{n+1}$ by showing that it is provable in $\mathsf{RCA}_0 + \mathsf{B}\Pi^0_n$. Jointly with Dorais, we show that $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_{n+1}$ proves $\mathsf{Elem\text{-}Indec}^n$ and also that $RT^1_2(\omega^3)$ is equivalent to ACA_0 over RCA_0.

The second theme of our study involves set theoretic forcing over models of RCA_0 and ACA_0. Our primary focus is on notions of forcing whose conditions are subtrees of $\omega^{<\omega}$ which are ordered by inclusion and have a simple property that we call "persistence". In his paper "A variant of Mathias forcing that preserves ACA_0", Dorais guides the reader through an interesting forcing construction. We use Dorais' framework and show that persistent notions of forcing over models of ACA_0 which satisfy a particular coloring property give rise to generic extensions which also model ACA_0. We also show that a slightly less restrictive property than persistence suffices to guarantee that generic extensions of models of RCA_0 are themselves models of RCA_0. Lastly, we work through several examples: Harrington, random, Sacks, Silver, and Miller forcing.