The Lagrangian and Relativistic Field Theory

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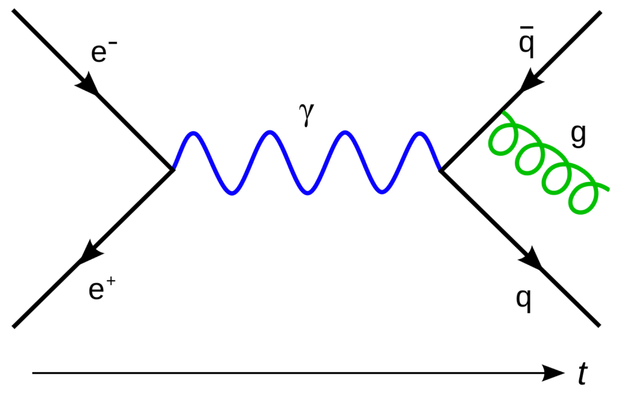

Abstract:

The Lagrangian is a formulism useful in deriving the equations motion for particles. In classical physics L is derived from L = T – U which is the difference between the kinetic and potential energy respectively. However in relativistic fields **L** is taken to be axiomatic, furthermore **L** is not unique for a particular system. Meaning multiplying and/or adding by a constant are all the same. The fields shown are all free meaning they have no sources or interactions. The Lagrangian is similar to Newton's equations, except it generalizes coordinates, time derivatives, and contains information about the dynamics of the system. The Lagrangian takes advantage of symmetries and geometric constraints. Symmetries are the mathematical structures that under pin the theories In the universe.

The Euler- Lagrange Equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_i{}'}\right) = \frac{\partial L}{\partial q_i} \qquad i = 1,2,3$$

q_i is a function of coordinates (q_1 = X, q_2 = Y, q_1 = Z)
$q_i{}'$ is their time derivatives ($q_1{}'$ = V_X, $q_2{}'$ = V_Y, $q_3{}'$ = V_Z)

In field theory, the major concern is calculating one or more functions of position and time. In order to be a relativistic theory the Lagrangian Density function **L** must treat space and time on equal footing.

Lagrangian Density Function *L* :

$$\partial_\mu \phi_i = \frac{\partial \phi_i}{\partial x^\mu}$$

Modified Lagrangian Density Function :

$$\partial_\mu \left(\frac{\partial \mathbf{L}}{\partial(\partial_\mu \phi_i)}\right) = \frac{\partial \mathbf{L}}{\partial \phi_i}$$

ϕ_i is a geometric field in a function of x, y, z, and t.

For a particle, a fermion, with spin ½ and mass m, L can be defined:

$$\mathbf{L} = i(\hbar c)\bar{\psi}\gamma^\mu\partial_\mu\bar{\psi} - (mc^2)\bar{\psi}\psi$$

Where ψ and $\bar{\psi}$ are treated as independent field variables and ψ is defined to be spinor field. ψ has dimensions of $L^{-3/2}$ which arises from Schrodinger's Wave Equation for a free fermion. The units of **L** is energy per unit volume.

The result is the following equation of motion:

$$\frac{\partial \mathbf{L}}{\partial(\partial_\mu \bar{\psi})} = 0$$

$$\frac{\partial \mathbf{L}}{\partial \psi} = i(\hbar c)\gamma^\mu\partial_\mu\psi - (\mathrm{mc}^2)\,\psi$$

$$i\gamma^\mu\partial_\mu\psi - (\mathrm{mc}/\hbar)\,\psi = 0$$

The Dirac Equation

This is the equation that describes the motion and dynamics of a particle with spin ½ and mass m in quantum field theory. This result is the Dirac Equation. If repeated for ψ instead of $\bar{\psi}$, the result is the adjoint of the Dirac Equation.

Proca Lagrangian for a Vector field

For a particle, a boson, with spin 1 and mass m, is defined as:

$$\mathbf{L} = -\frac{1}{16\pi}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{8\pi}(mc/\hbar)^2 A^\nu A_\nu$$

A^μ is a vector field and is defined as $\frac{\sqrt[2]{ML}}{T}$ and comes from Maxwell's Vector Potential

$$\frac{\partial \mathbf{L}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{4\pi}(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$\frac{\partial \mathbf{L}}{\partial A_\nu} = \frac{1}{4\pi}(mc/\hbar)^2 A^\nu$$

Thus this yields:

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) + (mc/\hbar)^2 A^\nu = 0$$

We can further simply by using the notation $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ and thus get:

$$\partial_\mu F^{\mu\nu} + (mc/\hbar)^2 A^\nu = 0$$

Notation:

The notation used in this formulation of the motion equations are derived from Electrodynamics and $F^{\mu\nu}$ is a fundamental quantity and the potentials are construed from it.

References

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